A Review of Apery's Theorem

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1 Introduction

The Riemann zeta function is the unique analytic continuation of the function $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ (a priori defined only on $s \in \mathbb{R} \times i\mathbb{R}_+$) to the whole complex plane.

This zeta function is one of the most (in)famous functions in mathematics, being the subject of the well-known **Riemann hypothesis**, one of the Millennium Problems. The zeta function pops up in many places, notably number theory.

Example 1.1 ([Sch76]). *The prime number theorem says that*

$$\pi(x) \sim li(x) = \int_0^x \frac{dt}{ln(t)} \sim \frac{x}{ln(x)}$$

where $\pi(x)$ is the prime counting function. Finding asymptotic behaviour of $\pi(x) - li(x)$ is harder. It is already known that

$$|\pi(x) - li(x)| = O(x^\beta log(x))$$

for some unknown $\frac{1}{2} \leq \beta \leq 1$. If the Riemann hypothesis is true then we have that

$$|\pi(x) - li(x)| = O(x^{\frac{1}{2}} \log(x)).$$

As this function is so important, it is natural to ask for properties of it. Of interest to us is the (ir)rationality of $\zeta(k)$ for $k \in \mathbb{N}$. Using analytic methods it is relatively straightforward to prove the following result.

Proposition 1.2 (Euler).

$$\zeta(2k) \in \pi^{2k}\mathbb{Q} \subsetneq \mathbb{R} \setminus \mathbb{Q}.$$

One might expect us to be able to show that $\zeta(2k+1) \in \pi^{2k+1}\mathbb{Q}$, but this has proven much harder. Very little progress was made until 1978, when Apery proved the following slightly weaker result.

Theorem 1.3 (Apery).

$$\zeta(3) \in \mathbb{R} \setminus \mathbb{Q}.$$

It is still unknown if $\zeta(3) \in \pi^3\mathbb{Q}$.

Apery's proof of the irrationality of this special value of $\zeta(2k+1)$ gave hope to the extending the methods used to other values of $\zeta(k)$. We will explain the proof of this result, and hopefully give some insight into why it has not been extended.

2 Apery's Proof

Apery's proof is quite bizarre and seems nearly magical; Poorten describes how "skepticism was general" during the initial presentation, and Beukers describes how "[his] surprise became excitement with Apery's announcement of his proof of $\zeta(3) \in \mathbb{R} \setminus \mathbb{Q}$ and ended in utter confusion after hearing his famous lecture" [Beu]. This is largely due to the strange methods and hard-to-see claims presented at each step. We will go over the proof, explaining details that are glossed over in Poorten's article on the subject.

We separate this proof into several steps. First we will introduce a result that will give us the irrationality of $\zeta(3)$, then construct a multivariate sequence that almost satisfies our conditions, and finally tweak our sequence to make it valid.

2.1 An irrationality Criterion

We first note a result on the irrationality of real numbers, heuristically saying "it is impossible to approximate a rational number too well with other rationals".

Lemma 2.1 (Dirichlet Irrationality Criterion)**.** *Let $\alpha \in \mathbb{R}$. If there exists a $\delta > 0$ and a sequence $\alpha \neq \frac{p_n}{q_n} \in \mathbb{Q}$ with $\frac{p_n}{q_n} \to \alpha$ such that*

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}} \qquad \textit{for all } n > N$$

for some $N \in \mathbb{N}$ then $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. Fix $\delta > 0$. We prove this using the contrapositive. Now let $\frac{a}{b}$ and $\frac{p}{q}$ be some rational numbers with $\frac{p}{q} \neq \frac{a}{b}$. Then we remark that

$$\left|\frac{a}{b} - \frac{p}{q}\right| = \frac{|aq - bp|}{|bq|} \geq \frac{1}{|bq|}.$$

Thus for any $r \in \mathbb{Q}$ we have that

$$\left|r - \frac{p}{q}\right| \geq \frac{1}{qq_0}$$

for some fixed q_0 that depends on r. For all $q^\delta > q_0$ we have that

$$\left|r - \frac{p}{q}\right| \geq \frac{1}{qq_0} > \frac{1}{q^{1+\delta}}$$

and so for a fixed $r \in \mathbb{Q}$ there are finitely many choices of q (all $q < q_0^{1/\delta}$) such that

$$\left| r - \frac{p}{q} \right| < \frac{1}{q^{1+\delta}}. \tag{1}$$

Noting that for the above to hold we must have that $p \in [qr - q^{-\delta}, qr + q^{-\delta}] \cap \mathbb{Z}$, it follows that there are at most finitely many $\frac{p}{q} \in \mathbb{Q}$ such that (1) holds. Thus if there exists a sequence $\frac{p_n}{q_n}$ satisfying the conditions of the theorem there are infinitely many rationals satisfying (1), and so we must have that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. $\square$

Thus to show a number α is irrational it suffices to find some sequence $r_n = \frac{p_n}{q_n}$ tending to α such that the **error** $\epsilon_n = |\alpha - r_n|$ shrinks faster than the **denominator** $q_n^{-1-\delta}$ shrinks.

2.2 Designing a Valid Sequence

We now start on a quest to find a suitable sequence. The first obvious candidate is the sequence of partial sums of $\zeta(3)$,

$$r_n = \sum_{k=1}^{n} \frac{1}{k^3}. \tag{2}$$

We can write

$$\left| \zeta(3) - \sum_{m=1}^{n} \frac{1}{m^3} \right| = \sum_{m=n+1}^{\infty} \frac{1}{m^3} = \epsilon_n$$

Now the denominator of r_n grows like $\text{lcm}(1, ..., n)^3$, and we have that

$$\frac{1}{2(n-1)^2} = \int_{n-1}^{\infty} \frac{dx}{x^3} > \sum_{m=n}^{\infty} \frac{1}{m^3} > \int_{n}^{\infty} \frac{dx}{x^3} = \frac{1}{2n^2}$$

and so ϵ_n shrinks like $\frac{1}{n^2}$. We will show that roughly $\text{lcm}(1, ..., n) \sim e^n$, and thus we have that

$$\epsilon_n > q_n^{-1-\delta}$$

for sufficiently large n, showing that this sequence does not converge fast enough to satisfy our requirements.

Lemma 2.2. $\text{lcm}(1, ..., n) \sim e^n$.

Sketch. We first note that for each prime $p \leq n$ then the highest factor of p that divides $\text{lcm}(1, ..., n)$ is the highest factor of p that occurs in the numbers $1, ..., n$. This is given by $\lfloor \log_p(n) \rfloor = \lfloor \frac{\ln n}{\ln p} \rfloor$. Then we can write

$$\text{lcm}(1, ..., n) = \prod_{p \leq n} p^{\lfloor \frac{\ln n}{\ln p} \rfloor}.$$

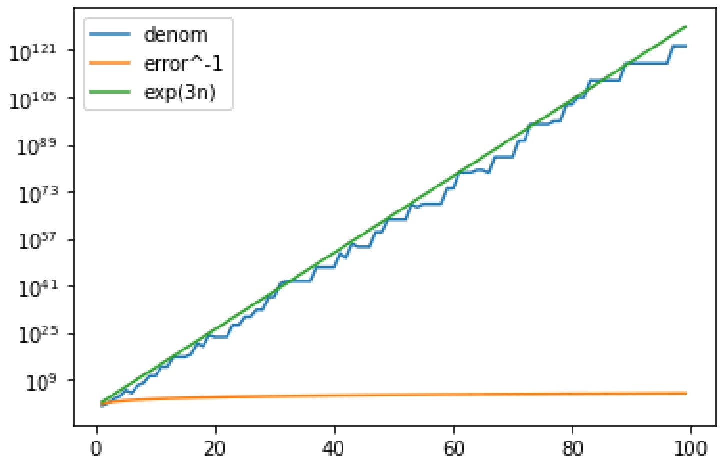

Figure 1: Error and denominator for sequence (2)

Taking logarithms of both sides gives

$$\begin{aligned}
\ln \operatorname{lcm}(1, ..., n) &= \prod_{p \le n} \left\lfloor \frac{\ln n}{\ln p} \right\rfloor \ln p \\
&\sim \prod_{p \le n} \frac{\ln n}{\ln p} \ln p \\
&= \prod_{p \le n} \ln n \\
&= \ln n \cdot \sum_{p \le n} 1 \\
&= \ln n \cdot \pi(n)
\end{aligned}$$

where $\pi(n)$ is the prime counting function. By the prime number theorem we have that $\ln n \cdot \pi(n) \sim n$ and so

$$\operatorname{lcm}(1, ..., n) \sim e^n.$$

□

We have thus shown that our candidate sequence r_n does not converge fast enough for us. The disparity is displayed in figure 1. We require the inverse error to asymptotically be greater than the denominator, but clearly this will not happen.

Indeed we must find a sequence that converges faster to $\zeta(3)$ while having relatively smaller denominators.

2.3 A Combinatorial Identity

In order to find a sequence that suits our purposes we turn to combinatorial identities for $\zeta(3)$. In particular Apery proved the identity

$$\zeta(3) = \frac{5}{2}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}. \tag{3}$$

The partial sums (denoted r_n) of this series also give a valid candidate for a valid sequence. Indeed we have (applying the fact that $\binom{2n}{n} \sim n^{-1/2}4^n$ from class) that

$$\begin{aligned}\left|\zeta(3) - r_n\right| &= \frac{5}{2}\sum_{k\geq n+1}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}} \\ &\leq \frac{5}{2}\frac{1}{\binom{2n}{n}}\sum_{k\geq n+1}\frac{1}{k^3} \\ &\sim \frac{5}{2}\frac{n^{1/2}}{4^n}\frac{1}{n^2} \\ &= \frac{5}{2}n^{-3/2}4^{-n}\end{aligned}$$

and so the error shrinks exponentially (instead of quadratically as before)! This sequence still does not satisfy our conditions. Indeed the denominator grows faster than $\text{lcm}(1, ..., n)^3 \sim e^{3n}$; since $e^3 > 4$, the denominator term $q_n^{-1-\delta}$ will shrink faster than ϵ_n. Nevertheless this direction is quite promising – our convergence has "sped up" in the right way, making the inverse error exponential instead of quadratic.

Instead we will modify (3) to try to get something that will work. We define the multivariate sequence

$$c_{n,k} = \sum_{i=1}^{n}\frac{1}{i^3} + \sum_{j=1}^{k}\frac{(-1)^{j-1}}{2j^3\binom{n}{j}\binom{n+j}{j}}. \tag{4}$$

This sequence, defined on the half-orthant defined by $\{(n,k) \in \mathbb{N}^2 \mid k \leq n\}$, is akin to a combined version of (2) and (3). It will turn out that this sequence will have properties similar to the fast convergence of (3), while the multi-dimensionality will be a useful tool to further tweak the sequence.

3 Behaviour of $c_{n,k}$

Having defined $c_{n,k}$ we give some important properties of it. We start by noting that $c_{n,k}$ is a rational number, however the exact form of the denominator is hard to pin down. The following lemma describes the nature of the denominator of $c_{n,k}$.

Lemma 3.1.

$$2\,\mathrm{lcm}(1,...,n)^3\binom{n+k}{k}c_{n,k}\in\mathbb{Z}.$$

The proof of this lemma is discussed in detail by Poorten in [PA79] and will prove useful later.

Lemma 3.2. *For any fixed $k = k_0$ we have that*

$$c_{n,k_0}\to\zeta(3)$$

as $n\to\infty$.

Proof. We have that

$$\begin{aligned}\left|\zeta(3)-c_{n,k_0}\right| &= \left|\sum_{i=n+1}^{\infty}\frac{1}{i^3}-\sum_{j=1}^{k_0}\frac{(-1)^{j-1}}{2j^3\binom{n}{j}\binom{n+j}{j}}\right|\\ &\le \sum_{i=n+1}^{\infty}\frac{1}{i^3}+\sum_{j=1}^{k_0}\frac{1}{2j^3\binom{n}{j}\binom{n+j}{j}}.\end{aligned}$$

We have already shown that the first term goes to 0 like $\frac{1}{n^2}$, and so it suffices to show that the second term goes to 0 as well. But we have that for $1\le j\le n$

$$2j^3\binom{n}{j}\binom{n+j}{j}\ge n^2$$

for large enough n, and so

$$\sum_{j=1}^{k_0}\frac{1}{2j^3\binom{n}{j}\binom{n+j}{j}}\le\sum_{j=1}^{k_0}\frac{1}{n^2}\le\frac{1}{n}.$$

where the last inequality uses $k_0\le n$. This shows that $c_{n,k_0}\to\zeta(3)$. □

Proposition 3.3. *For any direction $\mathbf{r}=(r_n,r_k)\in\mathbb{N}^2$ such that $(r_n,r_k)\in\{(n,k)\in\mathbb{N}^2\mid k\le n\}$ we have that*

$$c_{n\mathbf{r}}\to\zeta(3).$$

Proof. This is a result of the fact that the convergence of $c_{n,k}$ to $\zeta(3)$ does not depend on k, as shown in the proof of lemma 3.2. Strictly speaking we have that

$$\left|\zeta(3)-c_{n\mathbf{r}}\right|\le\frac{2}{nr_n}\le\frac{2}{n}$$

for large enough n. □

This proposition gives us an assortment of diagonal sequences that converge to $\zeta(3)$. Note that the diagonal sequence $c_{n,0}$ is the partial sums sequence given in (2). When investigating the error and denominators of this sequence it was found that the error decreased not nearly fast enough; let us instead check different diagonals of $c_{n,k}$ to see if they are promising.

3.1 $(1,1)$ Diagonal

We investigate the $(1,1)$ direction of $c_{n,k}$ to see if it satisfies our conditions. We first present two computational lemmas.

Remark 3.4. *We have that*

$$c_{n,k} - c_{n,k-1} = \frac{(-1)^{k-1}(k!)^2(n-k)!}{2k^3(n+k)!} \tag{5}$$

This follows from writing the definition for $c_{n,k}$ down.

Remark 3.5. *We have that*

$$c_{n,k} - c_{n-1,k} = \frac{(-1)^k k!^2(n-k-1)!}{n^2(n+k)!}. \tag{6}$$

Proof. This is described in [PA79] with typographical errors, which will be corrected here. From the definition of $c_{n,k}$ we have that

$$\begin{aligned}
c_{n,k} - c_{n-1,k} &= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3}\left[\frac{1}{\binom{n}{m}\binom{n+m}{m}} - \frac{1}{\binom{n-1}{m}\binom{n+m-1}{m}}\right] \\
&= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3}\left[\frac{m!^2(n-m)!}{(n+m)!} - \frac{m!^2(n-m-1)!}{(n+m-1)!}\right] \\
&= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}(m-1)!^2(n-m-1)!}{2m(n+m)!}\big[(n-m)-(n+m)\big] \\
&= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^{m}(m-1)!^2(n-m-1)!}{(n+m)!} \\
&= \frac{1}{n^3} + \sum_{m=1}^{k} \left[\frac{(-1)^m m!^2(n-m-1)!}{n^2(n+m)!}\right] - \left[\frac{(-1)^{m-1}(m-1)!^2(n-m)!}{n^2(n+m-1)!}\right] \\
&= \frac{1}{n^3} + \frac{(-1)^k k!^2(n-k-1)!}{n^2(n+k)!} - \frac{1}{n^3} \\
&= \frac{(-1)^k k!^2(n-k-1)!}{n^2(n+k)!}.
\end{aligned}$$

□

We can thus write

$$\begin{aligned}
c_{n,n} - c_{n-1,n-1} &= (c_{n,n} - c_{n,n-1}) + (c_{n,n-1} - c_{n-1,n-1}) \\
&= \frac{(-1)^{n-1}(n!)^2}{2n^3(2n)!} + \frac{(-1)^{n-1}(n-1)!^2}{n^2(2n-1)!} \\
&= \frac{5}{2}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}
\end{aligned}$$

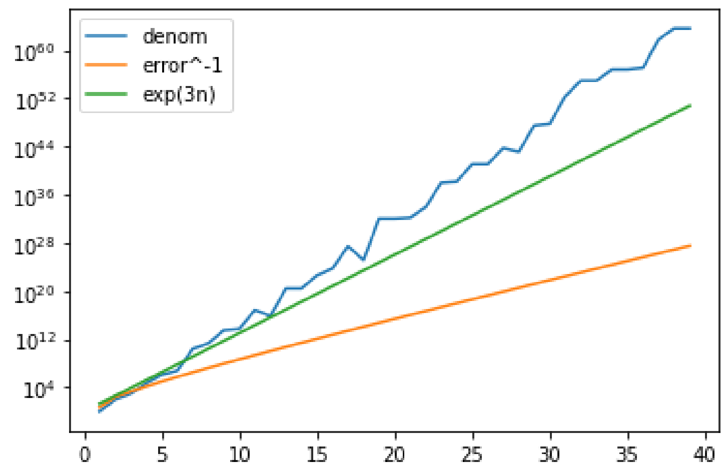

Figure 2: Error and denom for (4) in direction $(1,1)$

and so recalling our work on (3)

$$\left|\zeta(3) - c_{n,n}\right| = \frac{5}{2} \sum_{k \geq n+1} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}.$$

We have already shown that the exponential growth of this error is 4^{-n}, while the denominator grows like $(e^3 4)^n$, and so this diagonal still does not work.

Note that the sequence $c_{n,n}$ is designed such that the error is the tail end of the partial sums of the identity (3). This is no accident – it was defined in such a way as to maintain the rate of convergence of (3), while also allowing for manipulation dependent on the fact that it is a multidimensional sequence. As with (2) we capture the behaviour of $c_{n,k}$ in figure 2. One can see that the inverse error is exponential, but still grows slower than the denominator.

3.2 $(2,1)$ Diagonal

We can investigate the $(2,1)$ direction of $c_{n,k}$ to see if it satisfies our conditions.

By applying (5) and (6) we can write

$$\begin{aligned} c_{2n,n} - c_{2n-2,n-1} &= (c_{2n,n} - c_{2n,n-1}) \\ &\quad + (c_{2n,n-1} - c_{2n-1,n-1}) \\ &\quad + (c_{2n-1,n-1} - c_{2n-2,n-1}) \\ &= \frac{(-1)^n (n+1)!(n-1)!^5 n!^3}{16(n-1)^6 n^2 (3n)!(3n-1)!(3n-2)!}. \end{aligned}$$

It is no longer possible to represent this in terms of something like (3), as with the $(1,1)$ direction. We can however numerically estimate this, displayed in

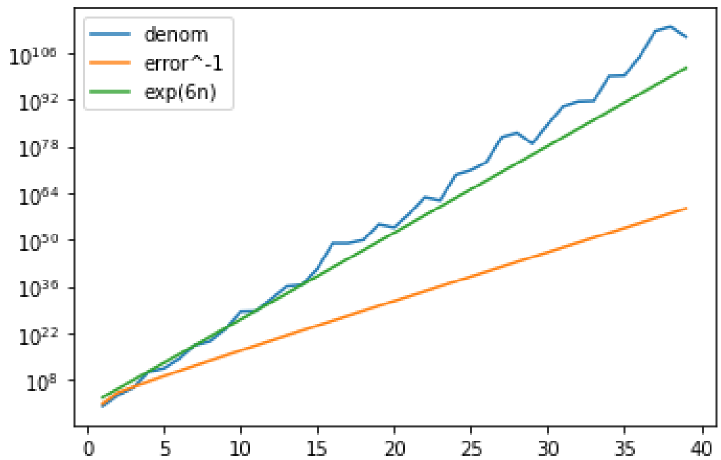

Figure 3: Error and denom for (4) in direction $(2, 1)$

figure 3 (we use $exp(6n)$ to compare, as $\text{lcm}(1, ..., 2n)^3 \sim e^{6n}$). See figures 5 and 6 for the directions $(5, 1)$ and $(10, 1)$ respectively. Numerically we see that there is little hope that $c_{2n,n}$ satisfies our requirements. In light of this we now turn to Apery's construction of a valid sequence.

4 Apery's Construction

We have seen evidence that $c_{n,k}$ does not converge fast enough for us. To remedy this, we transform $c_{n,k}$. Define

$$d_{n,k}^{(0)} = \binom{n+k}{k} c_{n,k},$$
$$w_{n,k}^{(0)} = \binom{n+k}{k}.$$

Note that $\frac{d^{(0)}_{n,k}}{w^{(0)}_{n,k}} = c_{n,k}$ and thus the quotient converges to $\zeta(3)$ along any diagonal. Now define successive transformations

$$
\begin{aligned}
d^{(1)}_{n,k} &= d^{(0)}_{n,n-k} & \qquad w^{(1)}_{n,k} &= w^{(0)}_{n,n-k} \\
d^{(2)}_{n,k} &= \binom{n}{k} d^{(1)}_{n,k} & \qquad w^{(2)}_{n,k} &= \binom{n}{k} w^{(1)}_{n,k} \\
d^{(3)}_{n,k} &= \sum_{l=0}^{k} \binom{k}{l} d^{(2)}_{n,l} & \qquad w^{(3)}_{n,k} &= \sum_{l=0}^{k} \binom{k}{l} w^{(2)}_{n,l} \\
d^{(4)}_{n,k} &= \binom{n}{k} d^{(3)}_{n,k} & \qquad w^{(4)}_{n,k} &= \binom{n}{k} w^{(3)}_{n,k} \\
d^{(5)}_{n,k} &= \sum_{l=0}^{k} \binom{k}{l} d^{(4)}_{n,l} & \qquad w^{(5)}_{n,k} &= \sum_{l=0}^{k} \binom{k}{l} w^{(4)}_{n,l}
\end{aligned}
$$

This appears quite strange, but turns out to accelerate the convergence of $c_{n,k}$. It is non-obvious but true that $\frac{d^{(i)}_{n,k}}{w^{(i)}_{n,k}}$ converges to $\zeta(3)$ along any diagonal.

Lemma 4.1. *For any direction* $\mathbf{r} \in \{(n,k) \in \mathbb{N}^2 \mid k \leq n\}$ *and any* $0 \leq i \leq 5$ *we have that*

$$\frac{d^{(i)}_{\mathbf{r}n}}{w^{(i)}_{\mathbf{r}n}} \to \zeta(3).$$

Sketch. Multiplying both $d^{(i)}_{n,l}$ and $w^{(i)}_{n,l}$ by the same constant does not change the convergence of the ratio. Changing (n,k) to $(n, n-k)$ amounts to changing the direction of the sequence from (an, bn) to $(an, (a-b)n)$. Since $c_{n,k}$ converges in any direction this operation will not change convergence. The only operation that we worry about will be taking $d^{(i)}_{n,k} = \sum_{l=0}^{k} \binom{k}{l} d^{(i-1)}_{n,l}$. But if $|\zeta(3) - \frac{d^{(i)}_{n,k}}{w^{(i)}_{n,k}}| \leq \epsilon_n$ for some $\epsilon_n \to 0$ then one can show that

$$\left| \zeta(3) - \frac{\sum_{l=0}^{k} \binom{k}{l} d^{(i)}_{n,k}}{\sum_{l=0}^{k} \binom{k}{l} w^{(i)}_{n,k}} \right| \leq \tilde{\epsilon}_n$$

where $\tilde{\epsilon}_n \to 0$ as $n \to \infty$. Thus $\frac{d^{(i)}_{\mathbf{r}n}}{w^{(i)}_{\mathbf{r}n}} \to \zeta(3)$ as $n \to \infty$. □

We have thus constructed a pair of multivariate sequences such that their quotient converges to $\zeta(3)$ in any direction. We define

$$
\begin{aligned}
a_n &= d^{(5)}_{n,n} \\
b_n &= w^{(5)}_{n,n} \in \mathbb{Z}
\end{aligned}
$$

to be the diagonals of our final sequences. It is clear that we maintain the property that

$$2 \operatorname{lcm}(1, ..., n)^3 d^{(i)}_{n,k} \in \mathbb{Z},$$

and thus we have that

$$2\,\mathrm{lcm}(1, ..., n)^3 a_n \in \mathbb{Z}. \tag{7}$$

It is true that $\frac{a_n}{b_n} \to \zeta(3)$. We will show that in fact this rational sequence satisfies our requirements.

4.1 Recurrence for a_n and b_n

It is miraculously true (and involved to show, consult [PA79] for a full derivation) that both a_n and b_n satisfy the recurrence given by

$$n^3 u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n-1)^3 u_{n-2} = 0 \tag{8}$$

for different initial values of u_0 and u_1 (we get a_n by setting $u_0 = 0, u_1 = 6$ and b_n by setting $u_0 = 1, u_1 = 5$).

We can immediately see that the dominant asymptotics of the sequence b_n are given by the recurrence

$$b'_n - 34b'_{n-1} + b'_{n-2} = 0$$

by taking the dominant terms of (8). Multiplying by x^n and summing over n gives us the generating function

$$B'(x) = \frac{1 + 39x}{1 - 34x + x^2}$$

for the asymptotic behaviour of b_n. The exponential nature of the dominant asymptotics of b_n are thus determined by the singularities $(1 \pm \sqrt{2})^4$ of the denominator. Since $b_n > 0$ and $(1-\sqrt{2}) < 0$ we have that the invisible coefficient in front of the term corresponding to $(1+\sqrt{2})^4$ cannot be 0. We thus have that (the asymptotics will also have some sub-exponential terms that will be unnecessary to note) the dominant exponential growth of b_n is given by $(1 + \sqrt{2})^4$.

4.2 Approximating ϵ_n

Our final step is now to use (8) to approximate ϵ_n asymptotically. Using (8) one can show that

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3},$$

and so by induction

$$\begin{aligned} \left|\zeta(3) - \frac{a_n}{b_n}\right| &= \sum_{k \geq n+1} \frac{6}{k^3 b_k b_{k-1}} \\ \text{(since } b_{n+1} > b_n \text{ for all } n) \quad &\leq \frac{1}{b_n^2} \sum_{k \geq n+1} \frac{6}{k^3} \\ &\leq b_n^{-2}. \end{aligned}$$

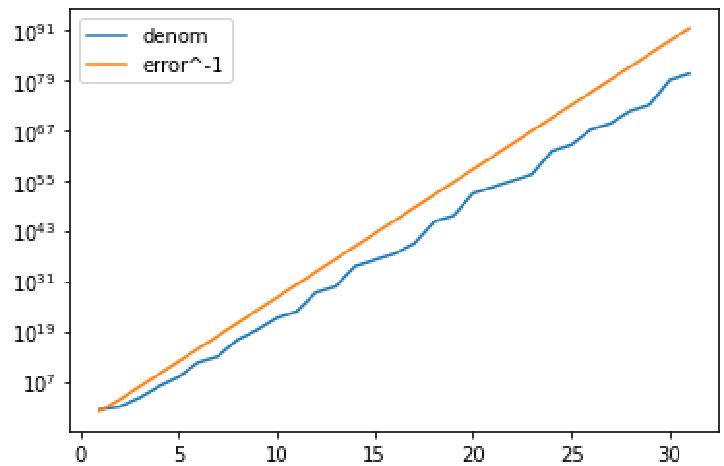

Figure 4: Error and denominator for (9)

Note that taking these inequalities (for sufficiently large n) will not affect exponential decay. Now we have almost constructed a sequence that works for us, since

$$b_n^{-2} < b_n^{-1-\delta}$$

and so ϵ_n decays exponentially faster than $b_n^{-1-\delta}$, as desired. However recall that a_n is not necessarily an integer, and so $\frac{a_n}{b_n}$ is not necessarily a valid rational approximation of $\zeta(3)$. To fix this we recall (7) and write

$$\left|\zeta(3) - \frac{2\,\mathrm{lcm}(1,...,n)^3 a_n}{2\,\mathrm{lcm}(1,...,n)^3 b_n}\right| \le b_n^{-2}. \tag{9}$$

Now the exponential asymptotic growth of the denominator of this rational approximation is $(e^3(1+\sqrt{2})^4)^n$, while the error has exponential asymptotic decay of $((1+\sqrt{2})^4 \cdot (1+\sqrt{2})^4)^{-n}$. Since $20 \approx e^3 < (1+\sqrt{2})^4 \approx 33$, this sequence satisfies the requirements of lemma 2.1 (with some explicitly calculable $\delta > 0$), we can conclude that $\zeta(3) \in \mathbb{R} \setminus \mathbb{Q}$. A visual representation is included in figure 4. Indeed we can see that now (and only now) the inverse error exceeds the denominator for large n.

4.3 Conclusion

The mysticality of this proof is apparent throughout; there is not much indication as to the construction of the sequence a_n/b_n would give a valid approximation of $\zeta(3)$, or that such a nice recurrence for a_n and b_n could be found. Perhaps hardest to digest is the fact that the bizarre construction described at the start of section 4 manages to speed up the convergence of $c_{n,n}$ from about decaying at an exponential rate of 4^{-1} to decaying at an exponential rate of $(1+\sqrt{2})^{-8}$ (a constant about 1/288.5 times the size) while only changing the growth of the denominator from $e^3 4$ to $e^3(1+\sqrt{2})^4$ (a constant about 8.5 times the size).

This said, it becomes hard to believe that we can apply proof to anything else. Indeed it has proven more difficult to extend this proof method to other values of $\zeta(2k+1)$ or proving irrationality of other constants, and no noteworthy results have been produced with the method since, interested readers are encouraged to consult [Coh] for a view into the search for recurrences similar to (8). A similar proof can be executed for $\zeta(2)$, also presented in Apery's paper, but both the proof for $\zeta(2)$ and $\zeta(3)$ rely on an identity of the form (8); the corresponding identity for $\zeta(2)$ is

$$\zeta(2) = 3 \sum_{n \geq 1} \frac{1}{n^2 \binom{2n}{n}}.$$

Both the $\zeta(2)$ and $\zeta(3)$ identity provide a rapidly-converging rational sequence of partial sums for ζ. If a similar identity was found for $\zeta(5)$ the proof for $\zeta(3)$ could likely be adapted, but as such no identity has been found, and it seems (after computer searching) that it is unlikely that a similar proof technique will work for $\zeta(5)$.

5 Appendix

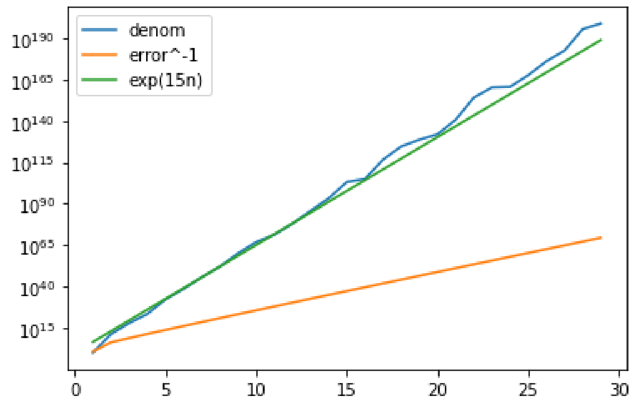

Figure 5: Error and denom for (4) in direction $(5, 1)$

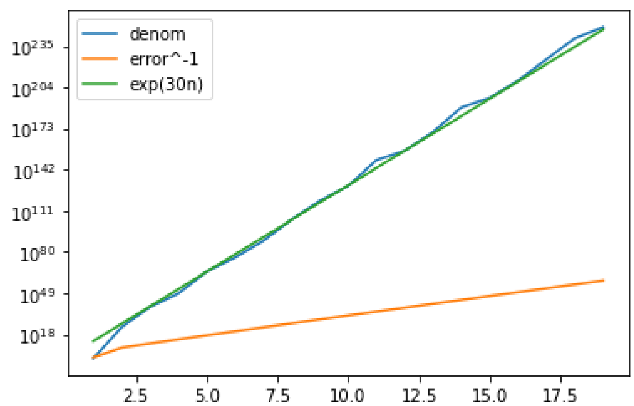

Figure 6: Error and denom for (4) in direction $(10, 1)$

References

[Beu] F. Beukers. “Consequences of Apery’s work on zeta(3)”. en. In: ().

[Coh] H. Cohen. “Apéry-Like Recursions and Modular Forms”. en. In: ().

[PA79] A. van der Poorten and R. Apéry. “A proof that Euler missed ...” en. In: *The Mathematical Intelligencer* 1.4 (Dec. 1979), pp. 195–203. ISSN: 0343-6993. DOI: 10.1007/BF03028234. URL: https://doi.org/10.1007/BF03028234 (visited on 04/15/2023).

[Sch76] L. Schoenfeld. “Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$. II”. In: *Mathematics of Computation* 30.134 (1976). Publisher: American Mathematical Society, pp. 337–360. ISSN: 0025-5718. DOI: 10.2307/2005976. URL: https://www.jstor.org/stable/2005976 (visited on 04/16/2023).