Donaldson Part 1

Alex Kroitor

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1 Introduction

Let X be a compact Riemann surface with a Hermitian metric, normalized to unit volume (the Hermitian metric gives a Kahler form, then we wedge it together a bunch to get a volume form, then that gives the volume of X, which we normalize), and let F be the associated curvature tensor (ie we get a canonical connection, for example Levi-Civita, then take the curvature of that). Let E be a vector bundle over X. Then define the slope as

$$\mu(E) := \frac{deg(E)}{rank(E)}$$

where

$$deg(E) = Tr(F) = \int \omega^{n-1} \wedge Tr(F).$$

The first Chern class does not depend on the connection picked, and so this is well-defined.

Definition 1.1 *We say that a holomoprhic vector bundle $\mathscr{E}$ is indecomposable if it cannot be written as a proper direct sum, ie*

$$\mathscr{E} \neq \mathscr{A} \oplus \mathscr{B}.$$

Definition 1.2 *We say that a holomorphic vector bundle $\mathscr{E}$ is stable if for all proper holomorphic sub-bundles $\mathscr{F} < \mathscr{E}$ we have*

$$\mu(\mathscr{F}) < \mu(\mathscr{E}).$$

Remark 1.3 *Any stable bundle is indecomposable. To see this let $\mathscr{E}$ be stable and assume we can write $\mathscr{E} = \mathscr{A} \oplus \mathscr{B}$. Then*

$$\mu(\mathscr{A}) < \mu(\mathscr{E}),$$
$$\mu(\mathscr{B}) < \mu(\mathscr{E}),$$

but

$$\mu(\mathscr{E}) = \frac{deg(\mathscr{A} \oplus \mathscr{B})}{rank(\mathscr{A} \oplus \mathscr{B})} = \frac{deg(\mathscr{A}) + deg(\mathscr{B})}{rank(\mathscr{A}) + rank(\mathscr{B})} < \mu(\mathscr{E})$$

by "mixing". The fact that $deg(\mathscr{A} \oplus \mathscr{B}) = deg(\mathscr{A}) + deg(\mathscr{B})$ follows from writing out $Tr(F)$ explicitly, and applying the additivity of Tr.

We would like to provide a converse statement.

Theorem 1.4 *An indecomposable (holomorphic) vector bundle $\mathscr{E}$ over X is stable iff there exists a unitary connection on $\mathscr{E}$ with constant central curvature*

$$*F = -2\pi i \mu(\mathscr{E}) I.$$

Such a connection is unique up to isomorphism.

Note 1.5 *F is a matrix of 2-forms, so the Hodge-star operation turns it into an element of $\Omega^0(End\mathscr{E})$ (a nice matrix of functions) by applying it component-wise.*

Remark 1.6 *It is 'easy' to prove equivalence of this result and the original one due to Narasimhan and Seshadri. Their proof supposedly relies on intense GIT and deformation theory.*

Remark 1.7 *If $deg(\mathscr{E}) = 0 = \mu(\mathscr{E})$ these connections are flat, and so given by unitary representations of the fundamental group. The idea here is that since the connection are flat, it only depends on the elements of the fundamental group (since parallel transport on a null-homotopic curve doesn't do anything).*

2.1 Definitions and Notation

If E is a C^∞ Hermitian vector bundle over X, a unitary connection A on E gives an operator

$$d_A : \Omega^0(E) \to \Omega^1(E).$$

This splits as

$$d_A = \partial_A + \overline{\partial}_A$$

where

$$\partial_A : \Omega^{0,0}(E) \to \Omega^{1,0}(E),$$
$$\overline{\partial}_A : \Omega^{0,0}(E) \to \Omega^{0,1}(E).$$

Here $\overline{\partial}_A$ is the Dolbeault operator corresponding to A, which defines a holomorphic structure $\mathscr{E}_A$ on E (by declaring a section σ to be holomorphic if $\overline{\partial}_A(\sigma) = 0$). Conversely if $\mathscr{E}$ is a holomorphic structure there's a unique (up to isomorphism) way to define a unitary connection A such that

$$\mathscr{E}_A = \mathscr{E},$$

and so we have a nice relation between holomorphic structures on E and unitary connections on E.

Let $\mathscr{A}$ be the space of unitary connections on E and let $\mathscr{G}$ (called the gauge group) be the group of unitary automorphisms of E (ie vector bundle automorphisms u such that $h(u(s), u(t)) = h(s, t)$). $\mathscr{G}$ acts (as a symmetry group, ie by conjugation) on $\mathscr{A}$ by

$$u(A) = A - (d_A u)u^{-1} \qquad u \in \mathscr{G},\ A \in \mathscr{A}.$$

Note that $u(A)$ is still unitary with respect to h. This action extends from $\mathscr{G}$ to the complexification $\mathscr{G}^{\mathbb{C}}$ – the group of general linear automorphisms of E – by

$$g(A) = A - (\overline{\partial}_A g)g^{-1} + ((\overline{\partial}_A g)g^{-1})^* \qquad g \in \mathscr{G}^{\mathbb{C}},\ A \in \mathscr{A}.$$

Note that $g(A)$ is not necessarily unitary with respect to the original metric h, but will be unitary with respect to a new metric (as an example, consider $\mathbb{C}^* = S^1 \times \mathbb{R}_{>0}$; the S^1 part is the unitary action and $\mathbb{R}_{>0}$ just 'rescales').

Remark 2.8 *Two connections define isomorphic holomorphic structures on E (here E is fixed, we are instead varying the holomorphic structure on E) precisely when they lie in the same $\mathscr{G}^{\mathbb{C}}$ orbit (the idea here is that if two holomorphic structures are isomorphic then there is a change of coordinates that sends one to the other; this is precisely the group $GL(C)$). Thus the set of orbits is the set of holomorphic vector bundles that have the same rank and degree as E (on a Riemann surface then a vector bundle E is characterized by its degree and rank, and so there are no bundles that sneak in; the justification is that a bundle can be decomposed into a line bundle and a trivial bundle, where the line bundle is determined by its Chern class).*

For any holomorphic bundle $\mathscr{E}$ we write $O(\mathscr{E})$ to denote the orbit of the associated connection. Finally we note that if A is a unitary connection then (for $a \in \Omega^1(End(E, h))$, where $End(E, h)$ has skew-Hermitian endomorphisms) $A + a$ is still a unitary connection. We compute

$$F(A + a) = F(A) + d_A(a) + a \wedge a. \tag{1}$$

2.2 Idea of Proof

The proof is by induction and is as follows.

- Assume that the result has been proven for bundles of lower rank. The rank 1 case follows 'easily' from Hodge theory.
- Define a functional J that takes in as input a connection A and spits out a positive real number. We choose a minimizing sequence of J in $O(\mathscr{E})$ and find a weakly converging subsequence (recall in a Hilbert space this is $H(a_n, b) \to H(a, b)$ means a_n weakly converges to a).
- Either the limit is $O(\mathscr{E})$ or is in another orbit $O(\mathscr{F})$. In the first case we show the result by taking small variations of the limit in $O(\mathscr{E})$. In the second case we apply the inductive hypothesis to achieve a contradiction.

We importantly use a result due to Uhlenbeck on the weak compactness of connections with L^2 bounded curvature.

Remark 2.9 *A weakly convergent sequence of C^∞ connections does not necessarily need to converge to a C^∞ connection. We will thus need to define connections of class L_1^2 (ie connections that differ from a a fixed C^∞ connection by an element of norm $||\alpha||_{L_1^2}^2 = ||\alpha||^2 + ||\nabla\alpha||^2$) with L^2 curvature and gauge transformations in L_2^2 (ie functions in $L_2^2(X, AutE)$). The group actions we defined extends painlessly (also $L_2^2 \hookrightarrow C^0$ and so the topology of the bundle is preserved). It is also true that each L_1^2 connection defines a holomorphic structure.*

2.3 Construction of Functional

Let M be a Hermitian matrix. We define

$$\nu(M) = Tr((M^*M)^{1/2}) = Tr((D^*D)^{1/2}) = \sum_{i=1}^{n} |\lambda_i|.$$

Note

- ν defines a norm.
- For $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$ then $\nu(M) \geq |Tr(A)| + |Tr(D)|$.

Now let s be a smooth self-adjoint section in $\Omega^0(EndE)$. Then define

$$N(s) = \left(\int_X \nu(s)^2 \right)^{1/2}$$

where here ν is being applied to each fibre of $\Omega^0(EndE)$ (this is functions from X to $EndE$; this is the case when the thing inside the brackets is a vector bundle). This norm is in fact equivalent to the L^2 norm. Now finally let A be an L_1^2 connection. Then we define

$$J(A) = N\left(\frac{*F(A)}{2\pi i} + \mu(\mathscr{E})I \right).$$

$J(A) = 0$ iff A has the desired properties! It is thus natural to look for a minimizer of J. We note that J is not smooth, but it does have the following property: if $A_i \to A$ weakly in L_1^2, so $F(A_i) \to F(A)$ weakly in L^2, then $J(A) \leq \liminf J(A_i)$.

Remark 2.10 *For bundles of rank 2 and degree 0 then J is more or less the Yang-Mills functional $||F||_{L^2}$. For larger ranks the definition of J is chosen to make the inductive step work nicely. The connections that we find in the end are Yang-Mills connections.*

3 Main Lemma

We take the following result for granted.

Proposition 3.1 *Let $A_i \in \mathscr{A}$ be a sequence of L_1^2 connections where $||F(A_i)||_{L^2}$ is bounded. Then there is a subsequence $\{i'\} \subset \{i\}$ and L_2^2 gauge transformations $u_{i'}$ such that $u_{i'}(A_{i'})$ converges weakly in L_1^2.*

Lemma 3.2 *Let $\mathscr{E}$ be a holomorphic vector bundle over X. Then either $\inf J|_{O(\mathscr{E})}$ is attained in $O(\mathscr{E})$, or there exists a holomorphic bundle $\mathscr{F} \not\simeq \mathscr{E}$ of the same degree and rank as $\mathscr{E}$ and with $\inf J|_{O(\mathscr{F})} \leq \inf J|_{O(\mathscr{E})}$; $Hom(\mathscr{E}, \mathscr{F}) \neq 0$.*

Proof. Pick a minimizing sequence A_i for $J|_{O(\mathscr{E})}$. $J(A_i)$ controls $N(F(A_i))$. Since N is equivalent to L^2, then $||F(A_i)||_{L^2}$ is bounded. Applying the above proposition gives WLOG that $A_i \to B$ weakly in L_1^2 (this should be up to a gauge transformation u_i, but it shouldn't matter), so $F(A_i) \to F(B)$ in L^2 (this is from the note after the definition of J). Then

$$J(B) \leq \liminf J(A_i) = \inf J|_{O(\mathscr{E})}.$$

B is an L_1^2 connection, and so it defines a holomorphic structure $\mathscr{E}_B$ on E. There are 2 possible cases.

- $\mathscr{E}_B \in O(\mathscr{E})$. Then $\mathscr{E}_B \simeq \mathscr{E}$, and so $J(B) = \inf J|_{O(\mathscr{E})}$ is attained in $O(\mathscr{E})$.
- $\mathscr{E} \notin O(\mathscr{E})$. Then $\mathscr{E}_B \not\simeq \mathscr{E}$. $\mathscr{E}_B$ has the same degree and rank as $\mathscr{E}$; it remains only to show that $Hom(\mathscr{E}_B, \mathscr{E}) \neq 0$.

Remark 3.3 (Call this something maybe) *Let $A, A' \in \mathscr{A}$ be two connections. Define a connection $d_{AA'}$ on the bundle $Hom(E, E) = E^* \otimes E$ built from the connection A on the left factor and A' on the right factor. This defines a Dolbeault operator*

$$\overline{\partial}_{AA'} : \Omega^{0,0}(Hom(E, E)) \to \Omega^{0,1}(Hom(E, E)).$$

We can write this as

$$\overline{\partial}_{AA'}(s) = \overline{\partial}_A(s) - s(\overline{\partial}_{A'}).$$

$\overline{\partial}_{AA'}$ vanishes exactly on holomorphic sections of $Hom(\mathscr{E}_A, \mathscr{E}_{A'})$.

Now suppose that $Hom(\mathscr{E}, \mathscr{E}_B) = 0$, and so $\overline{\partial}_{A_0 B}$ has trivial kernel. Then $\overline{\partial}_{A_0 B}$ is an elliptic operator of order 1 (a smooth connection is supposedly an elliptic operator of order 1, and the computations to show that $\overline{\partial}_{A_0 B}$ is also one are similar) with trivial kernel, and so

$$||\overline{\partial}_{A_0 B} s||_{L^2} \geq C||s||_{L_1^2} \geq C||s||_{L^4}$$

(in general if L is an elliptic operator of order k then $||u||_{H^{s+k}} \leq C||Lu||_{H^s}$ or something similar; see Folland for a proof). The second inequality follows from the Sobolev embedding theorem $L_1^2 \hookrightarrow L^4$ compactly.

Note since $L_1^2 \hookrightarrow L^4$ compactly, then $A_i \to B$ in the L^4 norm. We can calculate to find that

$$(\overline{\partial}_{A_0 B} - \overline{\partial}_{A_0 A_i})(s) = (\overline{\partial}_B - \overline{\partial}_{A_i})(s).$$

Then by Holder's inequality we have that

$$||(\overline{\partial}_B - \overline{\partial}_{A_i})(s)||_{L^2} \leq ||\overline{\partial}_B - \overline{\partial}_{A_i}||_{L^4}||s||_{L^4} \leq ||B - A_i||_{L^4}||s||_{L^4}$$

where the last inequality is just because A splits into ∂_A and $\overline{\partial}_A$ (note as well that where we apply Holder's inequality we actually apply it to the functions $|f|^2|g|^2$). Then by the triangle inequality we have

$$\begin{aligned} ||\overline{\partial}_{A_0 A_i} s||_2 &\geq ||\overline{\partial}_{A_0 B} s||_2 - ||(\overline{\partial}_{A_0 B} - \overline{\partial}_{A_0 A_i})s||_2 \\ &\geq C||s||_4 - ||B - A_i||_4||s||_4 \\ &= (C - ||B - A_i||_4)||s||_4. \end{aligned}$$

Since $A_i \to B$ in the L^4 norm, and so the right-hand side approaches $C||s||^4$, then we see that the kernel of $\overline{\partial}_{A_0 A_i}$ is trivial for large enough i. Hence $Hom(\mathscr{E}, \mathscr{E}_{A_i}) = 0$, contradicting $\mathscr{E}_{A_i} \simeq \mathscr{E}$.

4 What's Left?

It remains to show that:

- If the second case happens we achieve a contradiction. The idea for this is as follows:
- If the first case happens, ie that $\inf J|_{O(\mathscr{E})}$ is achieved in $O(\mathscr{E})$, then this minimum satisfies what we want.

4.1 Second Case

Assume that the second case of the lemma happens. It turns out that since $Hom(\mathscr{E}, \mathscr{F}) \neq 0$ we can draw

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{P} & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longleftarrow & \mathscr{N} & \longleftarrow & \mathscr{F} & \longleftarrow & \mathscr{M} & \longleftarrow & 0 \end{array}$$

$rank(\mathscr{Q}) = rank(\mathscr{M})$ and $deg(\mathscr{Q}) \leq deg(\mathscr{M})$.

We prove

1. $\inf J|_{O(\mathscr{F})} \geq J_0 := rank(\mathscr{M}) \cdot [\mu(\mathscr{M}) - \mu(\mathscr{F})] + rank(\mathscr{N}) \cdot [\mu(\mathscr{F}) - \mu(\mathscr{N})]$
2. $\inf J|_{O(\mathscr{E})} < J_1 := rank(\mathscr{P}) \cdot [\mu(\mathscr{E}) - \mu(\mathscr{P})] + rank(\mathscr{Q}) \cdot [\mu(\mathscr{Q}) - \mu(\mathscr{E})]$

We use the inductive hypothesis on $rank(\mathscr{E})$ to prove the second lemma. We also have $rank(\mathscr{Q}) = rank(\mathscr{M})$, $rank(\mathscr{P}) = rank(\mathscr{N})$, $deg(\mathscr{Q}) > deg(\mathscr{M})$, $deg(\mathscr{P}) < deg(\mathscr{N})$. Then we can show that

$$\inf J|_{O(\mathscr{E})} < J_1 \leq J_0 \leq \inf J|_{O(\mathscr{F})},$$

a contradiction to how we found such a $\mathscr{F}$. It follows that this second case can never happen, and so we always fall into the first case.

4.2 First Case

As part of the lemmas we have that if there is a connection A such that $*F(A) = -2\pi i \mu(\mathscr{E})$ (ie $J(A) = 0$) then $\mathscr{E}$ is in fact stable. For small t, pick a well-designed $g_t = 1 + th \in \mathscr{G}^{\mathbb{C}}$ (where here $d_A^* d_A h = 2\pi\mu(\mathscr{E}) - i * F(A)$). Then define

$$A_t = g_t(A) \in O(\mathscr{E}).$$

as a carefully chosen small variation of A. We can compute

$$N\left(\frac{*F(A_t)}{2\pi i} + \mu(\mathscr{E})\right) = N\left(\frac{*F(A)}{2\pi i} + \mu(\mathscr{E})\right)(1-t) + O(t^2).$$

If $J(A) \neq 0$ this yields a contradiction, since $J(A) \leq J(A_t)$ for all t. Thus we have that

$$*F(A) = -2\pi i \mu(\mathscr{E})$$

as desired.

Remark 4.1 *This is actually a very cool proof! A very cool theorem too.*