Preseminar Notes

You

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1 Introduction

These are notes for the pre-seminar. Thanks for coming!

1.1 Notation

We adopt the notation

- $(\underline{z}) = (z_1, \cdots, z_d)$
- $\underline{z}^{\underline{k}} = z_1^{k_1} \cdots z_d^{k_d}$
- $\overline{z} = 1/z$
- If $A(\underline{z}) = \sum_{\underline{i} \in \mathbb{Z}^d} f_{\underline{i}} \underline{z}^{\underline{i}}$ then $[\underline{z}^{\underline{n} \geq 0}]A(z) = \sum_{\underline{i} \in \mathbb{N}^d} f_{\underline{i}} \underline{z}^{\underline{i}}$
- If $A(\underline{z}) = \sum_{\underline{i} \in \mathbb{Z}^d} f_{\underline{i}} \underline{z}^{\underline{i}}$ then $\Delta A(z) = \sum_{n \geq 0} f_{n\underline{1}} t^n$

2 ACSV

2.1 Analytic Combinatorics

Enumerative combinatorics is the math of counting things. Often this is done by looking at a generating function. Let a_n be some sequence we care about.

$$\{a_n\}_{n \geq 0} \longleftrightarrow A(z) = \sum_{n \geq 0} a_n z^n.$$

This is traditionally **formal** (no evaluation of $\sum a_n z^n$ allowed!) Analytic combinatorics says *to heck with that* and treats the RHS as a normal sum (note this requires almost always that $\sum a_n z^n$ is convergent around 0). The natural setting for this is $\mathbb{C}$.

Example 2.1 *Consider $A(z) = \frac{1}{1+z^2}$. The radius of convergence is 1, but only problems are at $\pm i$.*

Enumerative combinatorics is often concerned with

$$\text{sequence} \longrightarrow GF.$$

Analytic combinatorics is more often concerned with

$$GF \longrightarrow \text{smtn about } a_n.$$

Typically the "something" is **asymptotics** of a_n.

3 Lattice Walks

The focus of this talk will be lattice walks.

Definition 3.1 (Lattice Walk Model) *A d-dimensional **lattice walk model** consists of:*

- *a (finite) **step set** $S \subseteq \mathbb{Z}^d \setminus 0$,*
- *a **restricting region** $R \subseteq \mathbb{Z}^d$,*
- *a **starting point** $p \in R$,*
- *a **terminal set** $T \subseteq R$.*

*A walk of length n in this lattice walk model consists of a tuple $(s_1, \cdots, s_n) \in S^n$ such that $p + s_1 + \cdots + s_j \in R$ for all $0 \leq j \leq n$ and $p + s_1 + \cdots + s_n \in T$. The **characteristic polynomial** of a lattice walk model is*

$$S(\underline{z}) = \sum_{s \in S} \underline{z}^{\underline{s}}.$$

Example 3.2 *EXAMPLE*

Note 3.3 *We will always let $p = 0$ and $T = R$. For mysterious technical reasons we restrict ourselves to **short-step** models, where $S \subseteq \{-1, 0, 1\}^d \setminus 0$.*

3.1 Dyck Paths

We will, as an example, find the generating function for Dyck prefixes.

Definition 3.4 *A **Dyck prefix** is a walk in the $1d$-lattice walk model $S = \{-1, 1\}, R = \mathbb{N}, p = 0, T = \mathbb{N}$.*

How do we find the GF for these? We can find a bijection from these to paths returning to the origin, but maybe we're lazy. Instead we proceed like a calculus student who's wandered into a combinatorics lecture.

Example 3.5 *Define*

$$F(x, t) = \sum_{n \geq 0} \left(\sum_{i \geq 0} f_{i,n} x^i \right) t^n$$

where $f_{i,n}$ is the number of Dyck prefixes of length n ending at i. Note that

- *$F(0, t)$ is the generating function for Dyck prefixes that end at the origin,*
- *$F(1, t)$ is the single-variate generating function for Dyck prefixes that does not track ending location.*

Then

$$\begin{aligned} F(x, t) &= 1 + (x + \overline{x})tF(x, t) - \overline{x}tF(0, t) \\ \implies (1 - t(x + \overline{x}))F(x, t) &= 1 - \overline{x}tF(0, t). \end{aligned}$$

Solving for $1 - t(x + \overline{x}) = 0$ for x gives

$$x = \frac{1 \pm \sqrt{1 - 4t^2}}{2t}.$$

Plugging this into the above gives

$$F(0, t) = \frac{1 - \sqrt{1 - 4t^2}}{2t^2}$$

(the + solution is absurd). Substituting this back into the original thing again and simplifying gives us

$$F(x,t) = \frac{1 - 2xt - \sqrt{1-4t^2}}{2t(t + tx^2 - x)}.$$

Thus

$$F(1,t) = \frac{1 - 2t - \sqrt{1-4t^2}}{2t(2t-1)} = \frac{1}{2t}\left(\sqrt{\frac{1+2t}{1-2t}} - 1\right)$$

as desired.

4 Kernel Method

What if we want to generalize this?

Question 4.1 *What is the GF for the 2d-lattice walk model $S = \{(-1,0),(1,0),(0,-1),(0,1)\}, R = \mathbb{N}^2, p = 0, T = \mathbb{N}^2$?*

To attack this we will use the **kernel method**. This is a method with two historical branches: probabilistic (attributed to Malyshev 1971) and combinatorial (attributed to Knuth 1968). Our modern incarnation of the method is used to solve certain systems of linear functional equations that seem to have "too many" unknowns (like $F(0,t)$ above). The naming comes from the idea that there is an important "kernel" $(1 - t(x + \overline{x})$ above) in the functional equations, which we may preserve while removing the unknowns. This will be shown by example.

We will demonstrate the kernel method for lattice walk models by using it on a small example.

Example 4.2 *Consider the lattice walk model given by $S = \{(-1,0),(1,0),(0,-1),(0,1)\}$. We would like to find the generating function for the number of walks of length n, starting at the origin, that remain in $\mathbb{N}^2$. Then*

$$S(x,y) = \overline{x} + x + \overline{y} + y.$$

We define a multivariate generating function that keeps track of end points

$$Q(x,y,t) = \sum_{n\geq 0}\left(\sum_{i,j\geq 0} q_{i,j,n}x^i y^j\right)t^n$$

where $q_{i,j,n}$ is the number of walks using S of length n that end at (i,j). Then note:

- *$Q(0,y,t)$ is the GF for walks ending on the x-axis,*
- *$Q(x,0,t)$ is the GF for walks ending on the y-axis.*

Then

$$\begin{aligned} Q(x,y,t) &= 1 + S(x,y)tQ(x,y,t) - \overline{y}tQ(x,0,t) - \overline{x}tQ(0,y,t) \\ \implies xy(1 - tS(x,y))Q(x,y,t) &= xy - xtQ(x,0,t) - ytQ(0,y,t). \end{aligned}$$

So far we have not done anything clever. We have three "unknowns" in this equation. We would really like to isolate $Q(x,y,t)$, but we can't. The genius comes from realizing the two involutions

$$\begin{aligned} \psi &: (x,y) \mapsto (\overline{x}, y), \\ \phi &: (x,y) \mapsto (x, \overline{y}) \end{aligned}$$

keep $S(x,y)$ fixed, and thus keep $K(x,y) = (1 - tS(x,y))$ fixed. The group generated (under composition) by ψ, ϕ is $\{id, \psi, \phi, \psi\circ\phi\}$. Then applying these 4 transformations to the above gives us the 4 equations

$$\begin{aligned} xyK(x,y)Q(x,y,t) &= xy - xtQ(x,0,t) - ytQ(0,y,t), \\ \overline{x}yK(x,y)Q(\overline{x},y,t) &= \overline{x}y - \overline{x}tQ(\overline{x},0,t) - ytQ(0,y,t), \\ x\overline{y}K(x,y)Q(x,\overline{y},t) &= x\overline{y} - xtQ(x,0,t) - \overline{y}tQ(0,\overline{y},t), \\ \overline{xy}K(x,y)Q(\overline{x},\overline{y},t) &= \overline{xy} - \overline{x}tQ(\overline{x},0,t) - \overline{y}tQ(0,\overline{y},t). \end{aligned}$$

We are maybe in a pickle, but we notice that each unknown term of the form $Q(x, 0, t)$ appears exactly twice. Taking a signed sum $(1) - (2) - (3) + (4)$ gives

$$xyQ(x, y, t) - \overline{x}yQ(\overline{x}, y, t) - x\overline{y}Q(x, \overline{y}, t) + \overline{xy}Q(\overline{x}, \overline{y}, t) = \frac{xy - \overline{x}y - x\overline{y} + \overline{xy}}{K(x, y)}.$$

But then each of the terms on the LHS except $xyQ(x, y, t)$ have at least one negative power of x or y. Thus

$$\begin{aligned} xyQ(x, y, t) &= [x^{n \geq 0} y^{n \geq 0}] \left(\frac{xy - \overline{x}y - x\overline{y} + \overline{xy}}{K(x, y)} \right) \\ \implies Q(x, y, t) &= [x^{n \geq 0} y^{n \geq 0}] \left(\frac{(x - \overline{x})(y - \overline{y})}{xy(1 - tS(x, y))} \right) \\ \implies Q(1, 1, t) &= \Delta \left(\frac{(1 + x)(1 + y)}{1 - txyS(x, y)} \right) \end{aligned}$$

and we have thus found the GF for these walks. Now how do we get the elements in these sequences?