Complex Monge-Ampère

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Contents

Chapter 0

Introduction

This is a collection of notes based on the class MAT1502HS (Topics in Geometric Analysis: Complex Monge-Ampère equations and Kähler geometry) taught by Freid Tong at the University of Toronto in Winter 2026.

The goal of this course will be to introduce Kähler geometry from the view of complex Monge-Ampère equations. We will take a broadly historical approach:

1. Yau's resolution of the Calabi conjecture, also known as Yau's theorem. This is a fundamental result on the Ricci curvature of Kähler manifolds. Yau's main contribution here was to prove a priori estimates for the complex Monge-Ampère.

2. Generalization of Yau's theorem to singular varieties. As before, this reduces to studying the complex Monge-Ampère. Here it turns out we need sharp a priori estimates.

3. Degenerate complex Monge-Ampère equations, and constructing geodesics in the space of Kähler metrics.

4. Additional topics.

0.1 Notation

We record our notation here.

To add. Bundle additive notation. Definition of ω_ϕ.

Notation (dfferentials). *We write interchangeably*

$$\partial_j = \frac{\partial}{\partial z_j},$$
$$\partial_{\bar{j}} = \frac{\partial}{\partial \bar{z}_j}.$$

Notation (Einstein notation). *We use Einstein notation, that is that (when unspecified) repeated lowered and raised indices are implicitly summed over:*

$$A_i B^i = \sum_i A_i B^i.$$

Notation (matrix inverses). *We use raised indices indicate the inverse of a matrix, so that*

$$A^{\mu\gamma} A_{\nu\gamma} = \delta^{\mu}{}_{\nu}.$$

Chapter 1

Basic Complex Geometry

Much of this should be review from Tristan's course last term.

1.1 Complex manifolds

Definition 1.1.1 (complex manifolds)**.** *A smooth manifold M is a **complex manifold** of (complex) dimension n (so real dimension $2n$) if and only if we can write $M = \bigcup_\alpha U_\alpha$ with maps*

$$\phi_\alpha : U_\alpha \to \mathbb{C}^n$$

such that its transition functions

$$\phi_\alpha \circ \phi_\beta^{-1}\Big|_{\phi_\beta(U_\alpha \cap U_\beta)} : \mathbb{C}^n \to \mathbb{C}^n$$

are holomorphic.

Given this data we can define holomorphic functions on M.

Definition 1.1.2 (holomorphic functions)**.** *We say a function $f : M \to \mathbb{C}$ is **holomorphic** if*

$$f \circ \phi_\alpha^{-1} : \mathbb{C}^n \to \mathbb{C}$$

is holomorphic for each α.

Remark 1.1.3. *This definition is okay since the transition maps are holomorphic. This collection of holomorphic functions determines the complex structure of M. With this definition, the coordinate charts $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ are holomorphic coordinate charts.*

Note that the local geometry of M is more or less the same as an open subset of $\mathbb{C}^n$. There are several important examples of complex manifolds.

Example 1.1.4 (complex projective space)**.** $\mathbb{P}^n = \{$*space of complex lines in* $\mathbb{C}^{n+1}\}$. *An element of $l \in \mathbb{P}^n$ has the form $l = [z_0 : \cdots : z_n]$, where $0 \neq (z_0, \ldots, z_n) \in l$. Here we identify $[z_0 : \cdots : z_n] \sim [\lambda z_0 : \cdots : \lambda z_n]$ for any $\lambda \in \mathbb{C}^*$. The coordinate charts are $U_i = \{[z_0 : \cdots : z_n] : z_i \neq 0\}$ for $i = 0, \ldots, n$. Here the chart maps are*

$$\begin{aligned} \phi_i : U_i &\to \mathbb{C}^n \\ [z_0 : \cdots : z_n] &\mapsto \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_n} \right). \end{aligned}$$

We can compute that the transition functions are

$$\begin{aligned} \phi_i \circ \phi_j^{-1}(w_1, \ldots, w_n) &= \phi_i([w_1 : \cdots : w_{j-1} : 1 : w_j : \cdots : w_n]) \\ &= \left(\frac{w_1}{w_i}, \ldots, \frac{\widehat{w_i}}{w_i}, \ldots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_j}{w_i}, \ldots, \frac{w_n}{w_i} \right), \end{aligned}$$

which are of course all holomorphic.

To make more complex manifolds we can take distinguished subsets of complex manifolds.

Theorem 1.1.5 (holomorphic implicit function theorem)**.** *Suppose we have a holomorphic function*

$$f : U \times V \subset \mathbb{C}^n_z \times \mathbb{C}^m_w \to \mathbb{C}^m$$

such that $f(0,0) = 0$ *and* $\det(\frac{\partial f}{\partial w})(0,0) \neq 0$*. Then there exists a* $g(z) : U' \subset U \to \mathbb{C}^m$ *holomorphic such that* $g(0) = 0$ *and* $f(z, g(z)) = 0$*.*

We can use this to get submanifolds of $\mathbb{P}^n$.

Example 1.1.6 (hypersurfaces in $\mathbb{P}^n$)**.** *Let* $f(z_0, \ldots, z_n) : \mathbb{C}^{n+1} \to \mathbb{C}$ *be a holomorphic homogenenous (degree* k*) polynomial. Then* $M = \{f = 0\} \subset \mathbb{P}^n$ *is a complex submanifold of dimension* $n-1$ *of* $\mathbb{P}^n$ *at all points* p *where* $f(p) = 0$ *and* $df(p) \neq 0$*. Repeatedly applying the holomorphic IFT gives us projective manifolds* $M^k \subset \mathbb{P}^n$ *of dimension* $k \leq n$*.*

Surely this depends on k.

1.1.1 Local structure of complex manifolds

We define

$$T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C},$$
$$\Omega_{\mathbb{C}}M = \Omega M \otimes_{\mathbb{R}} \mathbb{C}.$$

If M is a complex manifold, then in any holomorphic coordinate chart, $T_{\mathbb{C}}M$ is locally spanned by $\{\partial_i\}_{i=1}^n$ and $\{\partial_{\bar{i}}\}_{i=1}^n$ with complex coefficients. One can check that the subbundles

$$T^{1,0}M = \operatorname{span}_{\mathbb{C}} \{\partial_i\} \subset T_{\mathbb{C}}M,$$
$$T^{0,1}M = \operatorname{span}_{\mathbb{C}} \{\partial_{\bar{i}}\} \subset T_{\mathbb{C}}M,$$

are well-defined. This gives rise to a decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. Similarly we can define the subbundles

$$\Omega^{1,0}M = \operatorname{span}_{\mathbb{C}} \{dz_i\} \subset \Omega_{\mathbb{C}}M,$$
$$\Omega^{0,1}M = \operatorname{span}_{\mathbb{C}} \{d\bar{z}_i\} \subset \Omega_{\mathbb{C}}M,$$

and get the decomposition $\Omega_{\mathbb{C}}M = \Omega^{1,0}M \oplus \Omega^{0,1}M$. By taking wedge products we can define

$$\Omega^k_{\mathbb{C}}M = \bigoplus_{p+q=k} \Omega^{p,q}M$$

where $\Omega^{p,q}M$ is spanned locally by $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. The usual conjugation operation extends to $\Omega^{p,q}M \to \Omega^{q,p}M$ as

$$\overline{dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}} = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q},$$

and we say that $\alpha \in \Omega^k_{\mathbb{C}}M$ is real if $\alpha = \overline{\alpha}$. There is a differential structure on $\Omega^k M$ as well. We have

$$d : \Omega^k M \to \Omega^{k+1}M,$$

where if

$$\alpha = \sum_{|I|+|J|=k} \alpha_{I\overline{J}} dz_I \wedge d\bar{z}_J,$$
$$I = (i_1, \ldots, i_p),$$
$$J = (j_1, \ldots, j_q),$$
$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p},$$

then

$$d\alpha = \sum_{l=1}^{n} \sum_{|I|+|J|=k} \left(\frac{\partial \alpha_{I\bar{J}}}{\partial z_l} dz_l \wedge dz_I \wedge d\bar{z}_J + \frac{\partial \alpha_{I\bar{J}}}{\partial \bar{z}_l} d\bar{z}_l \wedge dz_I \wedge d\bar{z}_J \right).$$

From this we can see that in fact $d : \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1}$, and so we can write $d = \partial + \bar{\partial}$ (where ∂ is d projected onto the first term in this decomposition and $\bar{\partial}$ projects d onto the second term). As an exercise one can show that $\partial^2 = \bar{\partial}^2 = 0$ and so

$$0 = d^2 = (\partial + \bar{\partial})^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial.$$

Thus ∂ and $\bar{\partial}$ anticommute.

Definition 1.1.7 (ddbar operator)**.** *We define the* ***ddbar operator*** *to be the operator* $\sqrt{-1}\partial\bar{\partial}$*.*

Remark 1.1.8. *The factor of* $\sqrt{-1}$ *makes the ddbar operator a real operator.*

1.2 Holomorphic vector bundles

As one might expect, there is a notion of holomorphic vector bundles.

Definition 1.2.1 (holomorphic vector bundles)**.** *Let* $M = \bigcup_\alpha U_\alpha$ *be a complex manifold with a smooth complex vector bundle* $\pi : E \to M$ *of complex rank* r*. We say* E *is* ***holomorphic*** *if there exist trivializations* $\{e_{\alpha,\mu}\}_{\mu=1}^{r}$ *of* E *on* U_α *such that on* $U_\alpha \cap U_\beta$

$$e_{\alpha,\mu} = t_{\alpha\beta}{}^{\nu}{}_{\mu} e_{\beta,\nu}$$

for some holomorphic transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Mat}_{\mathbb{C}}(r \times r)$*.*

A holomorphic structure on a vector bundle his gives rise to an operator

$$\bar{\partial}_E : \Gamma(E) \to \Gamma(\Omega^{0,1} \otimes E),$$

where if $s = (s_\alpha)^\mu (e_\alpha)_\mu$ then

$$\bar{\partial}_E s = (\bar{\partial} s_\alpha{}^\mu) \otimes (e_\alpha)_\mu. \tag{1.1}$$

Note 1.2.2. *We often omit the* E *from* $\bar{\partial}_E$ *and write* $\bar{\partial}$*.*

A priori it is not clear that $\bar{\partial}$ is well-defined. On $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} \bar{\partial} s &= (\bar{\partial} s_\beta{}^\nu) \otimes (e_\beta)_\nu \\ &= \bar{\partial}(s_\alpha{}^\mu t_{\alpha\beta}{}^{\nu}{}_{\mu}) \otimes (e_\beta)_\nu \\ &= (\bar{\partial} s_\alpha{}^\mu) \otimes t_{\alpha\beta}{}^{\nu}{}_{\mu} e_{\beta,\nu} \\ &= (\bar{\partial} s_\alpha{}^\mu) \otimes e_{\alpha,\mu}, \end{aligned}$$

and so this operator $\bar{\partial}$ is invariant under the change of trivialization and is well-defined.

Note 1.2.3. *This does not work for* ∂*, as* $\partial t_{\alpha\beta}$ *is not necessarily* 0*.*

We can define the set of holomorphic sections as

$$H^0(M, E) = \{s \in \Gamma(M, E) : \bar{\partial} s = 0\} \subset \Gamma(M, E).$$

Note that the collection of holomorphic sections determines the holomorphic structure of E.

Note 1.2.4. *Also give a holomorphic vector bundle there exist local holomorphic trivializations* $\{e_{\alpha,\mu}\}$ *near any point. We always compute in these holomorphic trivializations.*

1.2.1 Associated bundles to E

Given a holomorphic vector bundle $E \to M$ there are several associated holomorphic vector bundles.

1. $E^* \to M$, where we replace $e_{\alpha\mu}$ by its dual trivialization $(e^*_\alpha)^\mu$ and the transition maps $t_{\alpha\beta}$ by $(t^{-1}_{\alpha\beta})^T$.

2. $\bigwedge^s E \to M$, with trivializations $e_{i_1} \wedge \cdots \wedge e_{i_s}$ and transition functions $\bigwedge^s t_{\alpha\beta} \in \mathrm{End}(\bigwedge^s E)$. If $s = r$ then $\bigwedge^s E$ is a holomorphic line bundle.

3. $\overline{E} \to M$ is an anti-holomorphic vector bundle with trivializations $\overline{e_{\alpha\beta}}$ and transition functions $\overline{t_{\alpha\beta}}$.

Example 1.2.5. *If M is a complex manifold, $T^{1,0}M$ and $\Omega^{1,0}M$ are both holomorphic vector bundles with trivializations $\{\partial_i\}_{i=1}^n$ and $\{dz_i\}_{i=1}^n$ respectively. The transition functions are as follow: if $(w_1, \dots, w_n)$ is another coordinate system then*

$$\frac{\partial}{\partial w_i} = \frac{\partial z_j}{\partial w_i}\frac{\partial}{\partial z_j} + \frac{\partial \bar{z}_j}{\partial w_i}\frac{\partial}{\partial \bar{z}_j} = \frac{\partial z_j}{\partial w_i}\frac{\partial}{\partial z_j}$$

and

$$dw_i = \frac{\partial w_i}{\partial z_j} dz_j.$$

Note here we are using the fact that the coordinate functions z_j are holomorphic.

Example 1.2.6. *The **canonical bundle** of M is $K_M = \bigwedge^n \Omega^{1,0}M = \Omega^{n,0}M$. This has trivialization $dz_1 \wedge \cdots \wedge dz_n$ and transition functions*

$$dw_1 \wedge \cdots \wedge dw_n = \det\left(\frac{\partial w}{\partial z}\right) dz_1 \wedge \cdots \wedge dz_n.$$

1.2.2 Hermitian metrics

A Hermitian metric on a bundle E ia s smoothly varying Hermitian inner product on the fibers of E. In a local holomorphic trivialization $\{e_\mu\}_{\mu=1}^r$ then

$$H = H_{\mu\bar{\nu}}(e^*)^\mu \otimes (\overline{e^*})^\nu \in E^* \otimes \overline{E}^*,$$

where $H_{\mu\bar{\nu}}$ is a positive-definite Hermitian matrix at each point. This gives an inner product on sections

$$\langle s, t\rangle_H = H_{\mu\bar{\nu}} s^\mu \overline{t}^\nu$$

which of course gives us a norm

$$|s|^2_H = \langle s, s\rangle_H = H_{\mu\bar{\nu}} s^\mu \overline{s}^\nu \geq 0.$$

A Hermitian metric H on E gives rise to a Hermitian structure on all associated bundles.

Example 1.2.7. *On E^* the induced metric is $\tilde{H} = H^{-1}$, so that*

$$\tilde{H} = H^{\mu\overline{\nu}} e_\mu \otimes \overline{e_\nu}.$$

Example 1.2.8. *On a line bundle any Hermitian metric is represented by a smoothly varying positive definite 1×1 matrix, and so the Hermitian metric is just a strictly positive smooth function h. In the specific case of the line bundle $\bigwedge^r E$, there is an induced Hermitian metric*

$$h = \det H_{\mu\bar{\nu}}.$$

Remark 1.2.9. *Hermitian metrics always exist by a partition of unity argument.*

We care about Hermitian metrics primarily because they can be used to define connections.

1.3 Connections

A connection is a way to differentiate sections of bundles to get more sections of bundles. Recall that we have a natural way to differentiate sections in anti-holomorphic directions (see (1.1)), but this naive approach to differentiating does not work for holomorphic directions (since the transition functions are not anti-holomorphic). To differentiate in holomorphic directions we must first pick a connection.

It turns out that if E is a Hermitian vector bundle (i.e., endowed with a Hermitian metric) then there exists a natural connection.

Definition 1.3.1 (Chern connection)**.** *Let E be a Hermitian vector bundle and pick a section $s = s^\mu e_\mu$. Then we define the **Chern connection** as the connection*

$$\nabla_{\bar{j}} s^\mu = \partial_{\bar{j}} s^\mu,$$
$$\nabla_j s^\mu = H^{\mu\overline{\nu}} \partial_j (H_{\gamma\overline{\nu}} s^\mu).$$

Note 1.3.2. *This looks a bit odd, but is actually natural.*

There is suerly a typo here. The index lowering operation here has indices that don't match up.

$$s^\mu \xrightarrow{\text{lower index}} H_{\gamma\overline{\nu}} s^\mu \xrightarrow{\partial_j \text{ well-defined}} \partial_j (H_{\gamma\overline{\nu}} s^\mu) \xrightarrow{\text{raise index}} H^{\mu\overline{\nu}} \partial_j (H_{\gamma\overline{\nu}} s^\mu).$$

Expanding out we see

$$\nabla_j s^\mu = \partial_j s^\mu + A_j{}^\mu{}_\nu s^\nu,$$

where $A_j{}^\mu{}_\nu = H^{\mu\overline{\gamma}} \partial_j H_{\nu\overline{\gamma}}$ are the associated connection coefficients. This definition induces the Chern connection on all tensor powers of E, E^*, and $\overline{E}$.

Example 1.3.3. *Consider E^* and a section $s = s_\mu (e^*)^\mu \in \Gamma(E^*)$. Then the Chern connection is*

$$\nabla_j s_\mu = \partial_j s_\mu - A_j{}^\nu{}_\mu s_\nu.$$

For tensor powers then ∇ gets applied to each component of the tensor product.

Example 1.3.4. *If $s = s^\mu{}_\nu (e^*)^\nu \otimes e_\mu \in \Gamma(E \otimes E^*)$ then one can show that*

$$\nabla_j s^\mu{}_\nu = \partial_j s^\mu{}_\nu + A_j{}^\mu{}_\gamma s^\gamma{}_\nu - A_j{}^\gamma{}_\nu s^\mu{}_\gamma.$$

1.3.1 Curvature

Once we have a connection we can define a notion of curvature. Unlike in Euclidean space, where all derivatives commute, this is not necessarily true for connections on an arbitrary vector bundle. The curvature associated to a specific connection measures the failure of ∇ to commute. We compute

$$\begin{aligned}
[\nabla_i, \nabla_{\bar{j}}] s^\mu &= \nabla_i \nabla_{\bar{j}} s^\mu - \nabla_{\bar{j}} \nabla_i s^\mu \\
&= \nabla_i (\bar{\partial}_{\bar{j}} s^\mu) - \nabla_{\bar{j}} (\partial_i s^\mu + A_i{}^\mu{}_\gamma s^\gamma) \\
&= \partial_i \partial_{\bar{j}} s^\mu + A_i{}^\mu{}_\gamma \partial_{\bar{j}} s^\gamma - \partial_{\bar{j}} \partial_i s^\mu - \partial_{\bar{j}} (A_i{}^\mu{}_\gamma s^\gamma) \\
&= -(\partial_{\bar{j}} A_i{}^\mu{}_\gamma) s^\gamma.
\end{aligned}$$

Definition 1.3.5 (curvature)**.** *In the spirit of the above computation, we define the **curvature** of a connection as*

$$F_{i\bar{j}}{}^\mu{}_\nu = -\partial_{\bar{j}} (A_i{}^\mu{}_\nu). \tag{1.2}$$

Exercise 1.3.6. *Check that $F_{ij} = F_{\bar{i}\bar{j}} = 0$, so that F is a (matrix of) $(1,1)$-forms.*

This formula extends to tensor powers.

Exercise 1.3.7. *Show that*

$$[\nabla_i, \nabla_{\bar{j}}] s^\mu{}_\nu = F_{i\bar{j}}{}^\mu{}_\gamma s^\gamma{}_\nu - F_{i\bar{j}}{}^\gamma{}_\nu s^\mu{}_\gamma$$

and that

$$[\nabla_i, \nabla_{\bar{j}}] \overline{s^\mu} = -\overline{F_{j\bar{i}}{}^\mu{}_\nu} \overline{s^\nu}.$$

1.3.2 Special Case of Line Bundles

Mostly we will be working with line bundles. In this case our formulas simplify significantly. If L is a holomorphic line bundle with a Hermitian metric h (locally we abuse notation slightly and write $h = he \otimes \bar{e}$) then

$$A_i = h^{-1}(\partial_i h) = \partial_i \log h$$

and

$$F_{i\bar{j}} = -\partial_{\bar{j}} A_i = -\partial_{\bar{j}} \partial_i \log h.$$

Then $F = \sqrt{-1} F_{i\bar{j}} dz^i \wedge d\bar{z}^j$ locally looks like

$$F = F_h = -\sqrt{-1}\partial\bar{\partial} \log h.$$

Note 1.3.8. *Note that F is not $\partial\bar{\partial}$ of a global function, as we cannot take a global trivialization of L to get a global h unless L is trivial.*

Exercise 1.3.9. *Check that $-\sqrt{-1}\partial\bar{\partial}\log h$ is a well-defined $(1,1)$-form, despite h not being globally defined.*

In general then F_h is a closed real $(1,1)$-form (as closedness is a local condition), but not an exact form (as h is not necessarily global). It makes sense then to think about the cohomology class of F_h.

Definition 1.3.10 (first Chern class)**.** *We define*

$$c_1(L) = [F_h] \in H^2(M, \mathbb{R})$$

*to be the **first Chern class** of L.*

It seems that $c_1(L)$ might depend on F_h, which depends on h, but this is not the case.

Theorem 1.3.11. *$c_1(L)$ is independent of h.*

Proof. Any other Hermitian metric is related to h by

$$\tilde{h} = e^{-\phi} h.$$

One can show as an exercise that such a ϕ is globally defined. Then

$$\begin{aligned} F_{\tilde{h}} &= -\sqrt{-1}\partial\bar{\partial}\log\tilde{h} \\ &= -\sqrt{-1}\partial\bar{\partial}\log(e^{-\phi}h) \\ &= F_h + \sqrt{-1}\partial\bar{\partial}\phi. \end{aligned}$$

Here then ϕ is a global function, and so $[\partial\bar{\partial}\phi] = 0$ (since $\partial\bar{\partial} = d\bar{\partial}$) and so the result follows. □

Example 1.3.12 (tautological bundle)**.** *Let $M = \mathbb{P}^n$. Each point of $\mathbb{P}^n$ corresponds to a line in $\mathbb{C}^{n+1}$. There is a natural line bundle on $\mathbb{P}^n$, called the **tautological bundle**, where the fiber above a point is the line corresponding to the point (ie the line going through any representative of the line):*

$$L = \mathcal{O}_{\mathbb{P}^n}(-1) = \{(l, v) : v \in l\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}.$$

As an exercise one can compute that the transition functions are $t_{ij}([z_0 : \cdots : z_n]) = \frac{z_j}{z_i}$, which are indeed holomorphic. One can also check (using the maximum principle) that there are no global holomorphic sections: $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)) = \{0\}$. The Euclidean metric on $\mathbb{C}^{n+1}$ induces a Hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(-1)$, where

$$|(z_0, \ldots, z_n)|_h^2 = \sum_{i=1}^n |z_i|^2,$$

and the curvature form associated to this metric is

$$F_h = -\sqrt{-1}\partial\bar{\partial}\log(\sum_{i=1}^n |z_i|^2).$$

Note 1.3.13. *This is almost not defined, since z_i is not well-defined, but the curvature is still well-defined.*

If you're on the chart $U_0 = \{z_0 = 1\}$ then

$$F_h = -\sqrt{-1}\partial\bar{\partial}\log(1 + |z_1|^2 + \cdots + |z_n|^2).$$

As a final exercise one can check that the curvature F_h with respect to the standard h is negative-definite.

Example 1.3.14 (canonical line bundle)**.** *If M is a complex manifold there's a distinguished line bundle*

$$K_M = \bigwedge^n(\Omega^{1,0}M) = \mathrm{span}_{\mathbb{C}}\{dz_1 \wedge \cdots \wedge dz_n\},$$

*called the **canonical line bundle**. We can take powers of K_M (with the convention that $K_M^{-1} = K_M^*$).*

Exercise 1.3.15 (tautological bundle and canonical bundle on $\mathbb{P}^n$)**.** *Check that on $\mathbb{P}^n$ with chart maps $\phi_j : U_j \to \mathbb{C}^n$ then if $(w_1', \ldots, w_n') = \phi_i \circ \phi_j^{-1}(w_1, \ldots, w_n)$ then*

$$dw_1' \wedge \cdots \wedge dw_n' = \left(\frac{z_j}{z_i}\right)^{n+1} dw_1 \wedge \cdots \wedge dw_n.$$

Then the transition maps of $K_{\mathbb{P}^n}$ are the same as the transition maps of $\mathcal{O}_{\mathbb{P}^n}(-1)$ to the power of $n+1$, and so (especially for $\mathbb{P}^n$) we have the relationship

$$K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1) = \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes(n+1)}.$$

1.4 Kahler Metrics

Let M be a complex manifold, and fix the holomorphic tangent bundle $T^{1,0}M$. A Hermitian metric g on M is a Hermitian metric $g = \sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ on $T^{1,0}M$ such that $\overline{g_{i\bar{j}}} = g_{j\bar{i}}$ and $g_{i\bar{j}}\zeta^i\bar{\zeta}^j \geq 0$. A Hermitian metric on M induces a Riemannian metric

$$g_R = \sum_{i,j} g_{i\bar{j}}(dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i),$$

and also induces a $(1,1)$-form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, where

$$\omega = \sqrt{-1}\sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Note that ω is real by the Hermitian condition.

Note 1.4.1. *All these objects carry the same data.*

Now a Riemannian metric induces a volume dV_g, and a $(1,1)$-form induces a volume form. One can check that $dV_g = \frac{\omega^n}{n!}$, and that locally

$$\frac{\omega^n}{n!} = (\det g_{i\bar{j}})(\sqrt{-1}dz^1 \wedge d\bar{z}^1) \wedge \cdots \wedge (\sqrt{-1}dz^n \wedge d\bar{z}^n).$$

Note 1.4.2. *The factors of $\sqrt{-1}$ here make everything real.*

Definition 1.4.3 (Kahler metric)**.** *We say that a Hermitian metric is **Kahler** if $d\omega = 0$.*

Note 1.4.4. *The conditions $(d\omega = 0)$, $(\partial\omega = 0)$, and $(\bar{\partial}\omega = 0)$ are all equivalent as ω is real.*

In local coordinates $(\partial\omega)_{ij\bar{k}} = \partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}}$, so

$$g \text{ Kahler } \iff \partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}. \tag{1.3}$$

Exercise 1.4.5 (Fubini-Study metric)**.** *Check that on $\mathbb{P}^n$ the metric*

$$\omega_{FS} = \sqrt{-1}\partial\bar{\partial}\log(|z_0|^2 + \cdots |z_n|^2)$$

*is Kahler. This is the **Fubini-Study metric** on $\mathbb{P}^n$. In the same way that $\mathbb{P}^n$ is the complex analogue of the sphere, the Fubini-Study metric is the analogue of the round metric on the sphere.*

Note 1.4.6. *In this course, unless otherwise stated, all manifolds are Kahler and compact. In particular submanifolds of Kahler manifolds are Kahler, so all projective manifolds are Kahler.*

1.5 Connection and Curvature of a Kahler Metric g

Consider a section $x = x^i \partial_i \in \Gamma(T^{1,0}M)$. Recall that we can write the Chern connection as

$$\nabla_{\bar{j}} x^i = \partial_{\bar{j}} x^i,$$
$$\nabla_j x^i = \partial_j x^i + \Gamma_j{}^i{}_k x^k.$$

Note 1.5.1. *Here the connection coefficients Γ are **Christoffel symbols**. They're usually written with A, but in this special case we use Γ.*

Recall from earlier that $\Gamma_i{}^j{}_k = g^{j\bar{l}} \partial_i g_{k\bar{l}}$. Then (1.3) implies that g is Kahler if and only if ∇ is **torsion-free**: $T_i{}^j{}_k = \Gamma_i{}^j{}_k - \Gamma_k{}^j{}_i = 0$.

Add reference.

Exercise 1.5.2. *Check that if $E \to M$ is a holomorphic vector bundle equipped with a Hermitian metric H, then $\nabla_i H_{\mu\bar{\nu}} = 0$.*

Note 1.5.3. *Recall that the Levi-Civita connection from Riemannian geometry is the unique connection that is metric compatible and torsion-free. The Chern connection is also metric compatible, and so in fact the two connections coincide if and only if g is Kahler.*

Let's compute the curvature of a Kahler manifold. From (1.2) we can write the **Riemannian curvature tensor** in local coordinates

$$\begin{aligned} R_{i\bar{j}k}{}^l &= -\partial_{\bar{j}} \Gamma_i{}^l{}_k \\ &= -\partial_{\bar{j}}(g^{l\bar{m}} \partial_i g_{k\bar{m}}) \\ &= -g^{l\bar{m}} \partial_{\bar{j}} \partial_i g_{k\bar{m}} \\ &= g^{p\bar{m}} g^{l\bar{q}} \partial_{\bar{j}} g_{p\bar{q}} \partial_i g_{k\bar{m}}. \end{aligned}$$

Here we're using Lemma A.1. Lowering the last index we can see

Say how.

There's an error in this computation somewhere I think.

$$R_{i\bar{j}k\bar{l}} = g_{m\bar{l}} R_{i\bar{j}k}{}^m = -\partial_i \partial_{\bar{j}} g_{k\bar{l}} + g^{p\bar{q}} \partial_{\bar{j}} g_{p\bar{l}} \partial_i g_{k\bar{q}}.$$

Since g is Kahler we can use (1.3) to check the following.

Lemma 1.5.4 (first Bianchi identity)**.** *In the above context we have that*

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{k\bar{l}i\bar{j}},$$

so that we can swap the two unbarred indices or the two barred indices. We can also take the complex conjugate to see that $\overline{R_{i\bar{j}k\bar{l}}} = R_{j\bar{i}k\bar{l}}$.

There is another similar identity that let us move around indices.

Lemma 1.5.5 (second Bianchi identity)**.** *In the above context we have that*

$$\nabla_i R_{p\bar{j}k\bar{l}} = \nabla_p R_{i\bar{j}k\bar{l}}.$$

Exercise 1.5.6. *Show that the second Bianchi identity holds.*

We can contract 2 indices of the Riemannian curvature tensor:

$$Ric = R_{i\bar{j}} = R_{i\bar{j}k}{}^k.$$

Note that here we are using the metric to raise one of the indices. By the symmetries in the Riemannian curvature tensor above, the result after contraction is unique, and we call it the **Ricci curvature tensor**. In the case of Kahler manifolds we get a particularly nice representation of the Ricci curvature tensor:

$$R_{i\bar{j}} = R_{i\bar{j}k}{}^k = -\partial_{\bar{j}}(g^{k\bar{m}} \partial_i g_{k\bar{m}}) = -\partial_{\bar{j}} \partial_i \log \det g_{k\bar{m}} \in c_1(-K_M). \tag{1.4}$$

This is because $\det g_{k\bar{m}}$ is a Hermitian metric on K_M^*, and so $R_{i\bar{j}}$ is of the form $-\partial\bar{\partial} \log h$. Here we're using Lemma A.2.

Note 1.5.7. *This is one of the main reasons why differential geometers care about the canonical bundle K_M: because it tells us about the Ricci curvature of a manifold.*

1.6 $\partial\bar{\partial}$-lemma

Given a line bundle $L \to M$ and a Hermitian metric h, we have seen that

$$F_h = -\sqrt{-1}\partial\bar{\partial}\log h \in c_1(L) \in H^2(M, \mathbb{R}),$$

where F_h is a real $(1,1)$-form. We can ask the inverse question: given a real $(1,1)$-form $\alpha = \sqrt{-1}\alpha_{i\bar{j}}dz^i \wedge d\bar{z}^j \in c_1(L)$, is α is the curvature of some Hermitian metric $\tilde{h}$? Any other metric can be written in the form $\tilde{h} = e^{-f}h$ so that

$$F_{\tilde{h}} = F_h + \sqrt{-1}\partial\bar{\partial}f.$$

If there's no such f, there cannot be such a $\tilde{h}$. It follows that we want to try to find an f satisfying the above. We cannot necessarily solve this, as two elements in $c_1(L)$ are related by a d-exact form, not a $\partial\bar{\partial}$-exact form. It turns out that with the extra condition of Kahlerity, this always works.

Theorem 1.6.1 ($\partial\bar{\partial}$-lemma)**.** *Let (M, ω) be a (compact) Kahler manifold. Then any d-closed (real) $(1,1)$-form $\alpha = \sqrt{-1}\alpha_{i\bar{j}}dz^i \wedge d\bar{z}^j$ can be written as $\alpha = \sqrt{-1}\partial\bar{\partial}f$ for some $f : M \to \mathbb{R}$.*

The idea for the proof comes from Hodge theory.

Fix this proof.

Proof. Let $\alpha = d\eta$ for some $\eta = \eta^{1,0} + \eta^{0,1}$. As α is real then $\eta^{0,1} = \overline{\eta^{1,0}}$. Then

$$\alpha = \partial\eta^{1,0} + \bar{\partial}\eta^{1,0} + \partial\overline{\eta^{1,0}} + \bar{\partial}\overline{\eta^{1,0}} = \bar{\partial}\eta^{1,0} + \partial\overline{\eta^{1,0}},$$

where the first and last terms vanish for degree reasons. Now write $\eta^{1,0} = \eta_k dz^k$. We want to solve the PDE

$$g^{k\bar{l}}\nabla_k\nabla_{\bar{l}}f = -g^{k\bar{l}}\nabla_{\bar{l}}\eta_k$$

for f. The LHS of this is $\Delta_g f$. Using Lemma A.4 it suffices to show that the integral of the RHS is 0, but

$$\int_M (-g^{k\bar{l}}\nabla_{\bar{l}}\eta_k)\omega^n = -\int_M \nabla_{\bar{l}}(g^{k\bar{l}}\eta_k)\omega^n = 0,$$

where here we are applying the divergence theorem and the fact that M has no boundary. Now

$$\partial(\eta^{1,0} + \partial f) = \partial^*(\eta^{1,0} + \partial f) = 0 \iff -g^{k\bar{l}}\nabla_{\bar{l}}(\eta_k + \partial_k f) = 0.$$

Claim a. *Let (M, ω) be compact Kahler. Suppose that $\beta = \beta_i dz^i$ satisfies*

1. *$\partial\beta = 0$ (or equivalently $\nabla_i\beta_j = \nabla_j\beta_i$),*
2. *$\partial^*\beta = 0$ (or equivalently $g^{k\bar{l}}\nabla_{\bar{l}}\beta_k = 0$).*

Then $\bar{\partial}\beta = 0$.

Given the claim we have that

$$\bar{\partial}\eta^{1,0} + \bar{\partial}\partial f = \bar{\partial}(\eta^{1,0} + \partial f) = 0,$$

and so

$$\alpha = \bar{\partial}\eta^{1,0} + \partial\overline{\eta^{1,0}} = 2\sqrt{-1}\partial\bar{\partial}\,\mathrm{im}(f).$$

It just remains to prove the claim.
Proof of Claim. We first note that

$$\begin{aligned}
\nabla_i\nabla_{\bar{j}}\beta_k &= \nabla_{\bar{j}}\nabla_i\beta_k - R_{i\bar{j}k}{}^l\beta_l \\
&= \nabla_{\bar{j}}\nabla_k\beta_i - R_{i\bar{j}k}{}^l\beta_l \\
&= \nabla_k\nabla_{\bar{j}}\beta_i + R_{k\bar{j}i}{}^l\beta_l - R_{i\bar{j}k}{}^l\beta_l \\
&= \nabla_k\nabla_{\bar{j}}\beta_i,
\end{aligned}$$

where here the second equality comes from the first condition we are assuming, and the fourth equality comes from the first Bianchi identity (Lemma 1.5.4). Then we compute that

Make sure that this is the correct notation for the norm.

$$
\begin{aligned}
||\bar{\partial}\beta||^2_{L^2(M,\omega)} &= \int_M |\bar{\partial}\beta|^2_\omega \omega^n \\
&= \int_M g^{k\bar{l}} g^{i\bar{j}} \partial_{\bar{j}}\beta_k \overline{\partial_{\bar{i}}\beta_l}\omega^n \\
&= \int_M g^{k\bar{l}} g^{i\bar{j}} \nabla_{\bar{j}}\beta_k \overline{\nabla_{\bar{i}}\beta_l}\omega^n \\
&= \int_M g^{k\bar{l}} g^{i\bar{j}} \nabla_{\bar{j}}\beta_k \nabla_i \overline{\beta_l}\omega^n \\
&= -\int_M \nabla_i \left(g^{k\bar{l}} g^{i\bar{j}} \nabla_{\bar{j}}\beta_k \right) \overline{\beta_l}\omega^n \\
&= -\int_M g^{k\bar{l}} g^{i\bar{j}} (\nabla_i \nabla_{\bar{j}}\beta_k) \overline{\beta_l}\omega^n \\
&= -\int_M g^{k\bar{l}} g^{i\bar{j}} (\nabla_k \nabla_{\bar{j}}\beta_i) \overline{\beta_l}\omega^n \\
&= -\int_M g^{k\bar{l}} \nabla_k \left(g^{i\bar{j}} \nabla_{\bar{j}}\beta_i \right) \overline{\beta_l}\omega^n \\
&= 0.
\end{aligned}
$$

Here equality 5 is IBP, equality 7 comes from the calculation above, equality 8 comes from the fact that $\nabla_k g^{i\bar{j}} = 0$ (Exercise 1.5.2), and equality 9 comes from the second condition that we assume. $\triangleleft$
We have shown the claim and so the result. $\square$

As a corollary to the $\partial\bar{\partial}$-lemma we have a positive result to the question at the beginning of the section.

Corollary 1.6.2. *Suppose that $L \to M$ is a holomorphic line bundle on a compact Kahler manifold. Then any real $(1,1)$-form $\alpha \in c_1(L)$ is the curvature of some Hermitian metric H.*

Chapter 2

The Calabi Conjecture

Let (X, ω) be a compact Kahler manifold. Recall from (1.4) that $[\mathrm{Ric}(\omega)] \in c_1(-K_M)$. To understand the Ricci curvature of Kahler manifolds, Calabi conjectured the following.

Think about what numbering I want here.

Conjecture 2.0.1 (Calabi Conjecture)**.** *Let $\alpha \in c_1(-K_M)$ be a real $(1,1)$-form. Then there exists a unique Kahler metric $\tilde{\omega}$ such that $[\tilde{\omega}] = [\omega]$ and $\mathrm{Ric}(\tilde{\omega}) = \alpha$.*

This was proven by Yau in 1978 (and so sometimes this result is called "Yau's Theorem"). Yau's proof 'started' Kahler geometry; before his proof Kahler geometry was a relatively niche field.

Remark 2.0.2. *One reason why this result is important is this gives us Ricci-flat manifolds: if K_M is trivial then $c_1(-K_M) = \{0\}$, so $\alpha = 0$. Then Yau's Theorem assures us that there exists a metric $\tilde{\omega}$ such that $\mathrm{Ric}(\tilde{\omega}) = 0$. This theorem also gives a strong connection between differential geometry (i.e.* Ric*, which is an analogue of the Laplacian showing up in the Einstein equations) and algebraic geometry (i.e. the study of manifolds with certain Chern classes).*

2.1 unpolished stuff below

Fix a real $(1,1)$-form $\alpha \in c_1(-K_M)$. Our road forward will be to reduce the problem to solving a complex Monge-Ampere equation. Fix ω. Then by the $\partial\bar{\partial}$-lemma (Lemma 1.6.1)

$$\alpha = \mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial} f$$

for some $f : M \to \mathbb{R}$. Moreover any other Kahler metric $\tilde{\omega}$ with $[\omega] = [\tilde{\omega}]$ can be written as

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\phi$$

for some $\phi : M \to \mathbb{R}$. We would like to find some $\tilde{\omega}$ such that $\mathrm{Ric}(\tilde{\omega}) = \alpha$, i.e. that

$$-\sqrt{-1}\partial\bar{\partial} \log\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\right) = \mathrm{Ric}(\tilde{\omega}) = \alpha = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}}) - \sqrt{-1}\partial\bar{\partial} f.$$

Rearranging this we get

$$-\sqrt{-1}\partial\bar{\partial}\left(\log\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i\partial\bar{z}_j}\right)}{\det(g_{i\bar{j}})} - f\right) = 0 \implies \det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial\bar{z}_j}\right) = e^{f+c}\det(g_{i\bar{j}}).$$

This is the **complex Monge-Ampere** equation. We usually rewrite this as

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{f+c}\omega^n.$$

Remark 2.1.1. *It turns out that c is determined by the data by integrating on both sides:*

Give more details on what is going on here, and also put the notation for

$$\int_M e^{f+c}\omega^n = \int_M (\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = \int_M \omega\wedge\omega_\phi^{n-1} + \int_M d(\bar{\partial}\phi\wedge\omega_\phi^{n-1}) = \int_M \omega\wedge\omega_\phi^{n-1} = \cdots = \int_M \omega^n.$$

It follows that $c = \log\left[\frac{\int_M \omega^n}{\int_M e^f \omega^n}\right]$*. Here we absorb* c *into* f*, but must keep track of the fact that* $\int_M e^f\omega^n = \int_M \omega^n$*. This leads to the most popular formulation of the complex Monge-Ampere:*

$$\omega_\phi^n = e^f \omega^n. \tag{2.1}$$

Theorem 2.1.2 (Yau 1978)**.** *Given* $f : M \to \mathbb{R}$ *such that* $\int_M e^f\omega^n = \int_M \omega^n$*. Then there exists* $\phi : M \to \mathbb{R}$*, unique up to adding a constant, such that* $\omega_\phi > 0$ *and* $\omega_\phi^n = e^f\omega^n$*.*

Calabi proved uniqueness (which is not too difficult), while Yau proved existence (which was hard enough that it gave him a Fields Medal).

Proof (of uniqueness). Let's say we have two solutions

$$\begin{aligned}\omega_1 &= \omega + \sqrt{-1}\partial\bar\partial\phi_1,\\ \omega_2 &= \omega + \sqrt{-1}\partial\bar\partial\phi_2 = \omega_1 + \sqrt{-1}\partial\bar\partial\phi\end{aligned}$$

for $\phi = \phi_1 - \phi_2$. Now $\omega_1^n = \omega_2^n = (\omega_1 + \sqrt{-1}\partial\bar\partial\phi)^n$ by (2.1). Now integrate this to get

Check subscript on norm in second-last line.

$$\begin{aligned}0 &= -\int_M \phi(\omega_2^n - \omega_1^n)\\ &= -\int_M \phi\left[(\omega_1 + \sqrt{-1}\partial\bar\partial\phi)^n - \omega_1^n\right]\\ &= -\int_M \sqrt{-1}\phi\,\partial\bar\partial\phi \wedge (\omega_2^{n-1} + \omega_2^{n-2}\wedge\omega_1 + \cdots + \omega_2\wedge\omega_1^{n-2} + \omega_1^{n-1})\\ &= -\int_M \sqrt{-1}\phi\, d\bar\partial\phi \wedge (\omega_2^{n-1} + \omega_2^{n-2}\wedge\omega_1 + \cdots + \omega_2\wedge\omega_1^{n-2} + \omega_1^{n-1})\\ &= -\int_M \sqrt{-1}\phi\, d\big[\bar\partial\phi \wedge (\omega_2^{n-1} + \omega_2^{n-2}\wedge\omega_1 + \cdots + \omega_2\wedge\omega_1^{n-2} + \omega_1^{n-1})\big]\\ &= \int_M \sqrt{-1}d\phi\wedge\bar\partial\phi \wedge (\omega_2^{n-1} + \omega_2^{n-2}\wedge\omega_1 + \cdots + \omega_2\wedge\omega_1^{n-2} + \omega_1^{n-1})\\ &= \int_M \sqrt{-1}\partial\phi\wedge\bar\partial\phi \wedge (\omega_2^{n-1} + \omega_2^{n-2}\wedge\omega_1 + \cdots + \omega_2\wedge\omega_1^{n-2} + \omega_1^{n-1})\\ &\geq \int_M \sqrt{-1}\partial\phi\wedge\bar\partial\phi\wedge\omega_1^{n-1}\\ &= \frac{1}{n}\int_M |\partial\phi|_\omega^2\omega_1^n,\end{aligned}$$

and so ϕ is a constant. In the above computation equality 6 is IBP, equality 7 is using the fact that $\bar\partial\phi\wedge\bar\partial\phi = 0$, equality 8 is using the fact that the integrand has a positive term times a sum of wedge powers of positive forms (which is positive by Lemma A.6). We are also using Lemma A.5 in equality 3 and Lemma A.7 in equality 9. □

2.2 Existence: Method of Continuity

To prove the existence of a solution to (2.1) we use the *method of continuity*. The idea is as follows. Consider the following family of PDEs, depending on $t \in [0, 1]$:

Fiddle with the formatting in this environment.

$$\begin{cases}(\omega + \sqrt{-1}\partial\bar\partial\phi_t)^n & = e^{tf+c_t}\omega^n,\\ \omega + \sqrt{-1}\partial\bar\partial\phi_t & > 0,\end{cases} \tag{$*_t$}$$

where c_t is chosen such that $\int_M e^{tf+c_t}\omega^n = \int_M \omega^n$. When $t = 1$ then $(*_t)$ is the equation we want to solve. When $t = 0$ then $(*_t)$ is trivial to solve (with $\phi = 0$). Then define the set

$$I = \{t \in [0, 1] : (*_t) \text{ has a solution}\}.$$

If we can show that I is both open and closed then $I = [0, 1]$ (as $0 \in I \neq \emptyset$), and so we are done. Showing these use different tools:

1. Openness follows from an application of IFT.
2. Closedness follows from an application of a priori estimates.

There's a blurb here, in lec4 page 15, on a priori estimates. This is repeated later. Maybe include it here if it's suitable.

2.2.1 Openness

Let's prove openness. Recall the following version of IFT.

Maybe put this in Appendix, but probably not

Theorem 2.2.1 (Banach space IFT)**.** *Let B_1, B_2, B_3 be Banach spaces. Suppose that*

$$F : U_x \times V_y \subset B_1 \times B_2 \to B_3$$

is a C^1-Fréchet differentiable. If $F(0,0) = 0$ and $\frac{\partial F}{\partial y}$ is invertible with bounded inverse, then there exists a neighbourhood $\tilde{U} \subset B_1$ of 0 and $g : \tilde{U} \to B_2$ such that

$$F(x, g(x)) = 0.$$

Let's use this to prove openness. Let $B_1 = \mathbb{R}_t$ and B_2, B_3 be two function spaces that are to be determined. Define the function

$$F(t, \phi) = \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\omega^n} - e^{tf + c_t}. \tag{2.2}$$

Note that ϕ_t solving $(*_t)$ is equivalent to $F(t, \phi_t) = 0$. Clearly $F(0,0) = 0$. To apply IFT we must show that $D_\phi F(t, \phi_t)$ is bounded with bounded inverse (we will pick B_2, B_3 to ensure this). Now if ϕ_t solves $(*_t)$ then

Make sure this calculation makes sense. In particular the $\tilde{g}$ part.

$$\begin{aligned}
(D_\phi F(t, \phi_t))u &= \frac{d}{ds}\bigg|_{s=0} F(t, \phi_t + su) \\
&= \frac{d}{ds}\bigg|_{s=0} \left[\frac{(\omega + \sqrt{-1}\partial\bar{\partial}(\phi_t + su))^n}{\omega^n} - e^{tf + c_t} \right] \\
&= \frac{1}{\det g_{i\bar{j}}} \frac{d}{ds}\bigg|_{s=0} \det\left(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\phi + s\partial_i\partial_{\bar{j}}u\right) \\
&= \frac{\det(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\phi)}{\det g_{i\bar{j}}} \tilde{g}^{i\bar{j}}\partial_i\partial_{\bar{j}}u,
\end{aligned}$$

where $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\phi$. Here we are applying Lemma A.3. Thus

$$(D_\phi F(t, \phi_t))u = \frac{\omega_\phi^n}{\omega^n}\Delta\omega_\phi u.$$

We want to pick B_2, B_3 such that this is invertible. This is more or less just a Laplacian, so we can use our knowledge of elliptic PDEs. Recall from Lemma A.4 that the equation $\Delta u = f$ is solvable if and only if $\int_M f dV_g = 0$. This suggests that B_2, B_3 should have the condition $\int_M f dV_g = 0$ baked into it. Schauder theory (i.e. elliptic PDE theory) gives us that

$$\Delta : C_0^{2,\alpha}(M) \to C_0^{\alpha}(M)$$

is invertible with bounded inverse, where

$$C_0^{2,\alpha}(M) = C_0^{2,\alpha}(M, g) = C^{2,\alpha}(M) \cap \{\int_M f = 0\}.$$

Note 2.2.2. *If $\int_M f\omega^n = 0$ then $\int_M f\frac{\omega^n}{\omega_\phi^n}\omega_\phi^n = 0$.*

Say more here.

Then we naturally pick

$$\begin{aligned}
B_2 &= C_0^{2,\alpha}(M, \omega), \\
B_3 &= C_0^{\alpha}(M, \omega).
\end{aligned}$$

Exercise 2.2.3. *Verify that $u \mapsto \frac{\omega_\phi^n}{\omega^n}\Delta_{\omega_\phi}u$ is indeed the Fréchet derivative of F, and check that it's actually C^1.*

Note 2.2.4. *Generally Holder spaces are a better setting for non-linear PDEs, as the product of two $C^{k,\alpha}$ functions is once again $C^{k,\alpha}$, but $W^{k,p}$ is not closed under multiplication.*

2.3 Closedness

Closedness is the key input. Suppose that $t_j \to t_\infty$ and ϕ_j solves $(*_t)$ for $t = t_j$. Then we would like to take some limit of ϕ_j to a solution of $(*_t)$ for $t = t_\infty$. For the equation $(*_t)$ to converge we require C^2 convergence (as on the LHS we have two derivatives on the LHS). To take this limit we need some form of compactness, but C^2 isn't good enough to extract a subsequence, and instead we need slightly better convergence to be able to apply Arzela-Ascoli. We thus need some bound of the form

Fix the labelling issues here.

$$||\phi_j||_{C^{2,\alpha}} \leq C$$

independent of j. This requires an *a priori estimate* for solutions of (2.1), i.e. bounds of the form $||\phi||_{C^{2,\alpha}} \leq C$, where C depends on the background data. There are typically 4 steps when it comes to showing regularity.

1. $||\phi||_{C^0} \leq C$.
2. $||D\phi||_{C^0} \leq C$.
3. $||D^2\phi||_{C^0} \leq C$.
4. $||\phi||_{C^3} \leq C$.

Once we have a C^3 bound, Schauder theory gives us a bound

$$||\phi||_{C^{k,\alpha}} \in C_{k,\alpha}$$

for all k, α. Historically blah blah.

Add historial note here

Remark 2.3.1. *For finding Kahler-Einstein metrics there's a similar setup. All estimates work except for the C^0 estimate. Then the existence of the C^0 estimate is somehow related to the geometry of the situation.*

We'll start with the precursor to the C^0 estimate.

2.3.1 L^2 Estimate

We want to show that if ϕ solves (2.1) then we get a bound

$$\int_M |\phi|^2\omega^n \leq C = C(||f||).$$

Remark that we must normalize ϕ to have any hope of having an L^2 bound for ϕ. Normalize ϕ now so that $\int_M \phi\omega^n = 0$. Recall as well the following bound.

Maybe put in Appendix.

Lemma 2.3.2 (Poincaré inequality)**.** *Let (M, g) be a compact Riemannian manifold. Then there exists a $C > 0$ such that for any $f : M \to \mathbb{R}$ with $\int_M f dV_g = 0$ then*

$$\int_M f^2 dV_g \leq C \int_M |\nabla f|_g^2 dV_g.$$

Now rewrite (2.1) as

$$\omega_\phi^n - \omega^n = (e^f - 1)\omega^n.$$

Multiply this by $-\phi$ and integrate to get

$$\begin{aligned}
-\int_M \phi(e^f - 1)\omega^n &= -\int_M \phi(\omega_\phi^n - \omega^n) \\
&= -\int_M \phi\sqrt{-1}\partial\bar{\partial}\phi \wedge (\omega_\phi^{n-1} + \omega_\phi^{n-2} \wedge \omega + \cdots + \omega_\phi \wedge \omega^{n-2} + \omega^{n-1}) \\
&= \int_M \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge (\omega_\phi^{n-1} + \omega_\phi^{n-2} \wedge \omega + \cdots + \omega_\phi \wedge \omega^{n-2} + \omega^{n-1}) \\
&\geq \frac{1}{n}\int_M |\nabla\phi|^2\omega^n \\
&\geq \frac{1}{C}\int_M \phi^2\omega^n.
\end{aligned}$$

Here we are applying more or less the same ideas we used to prove uniqueness in Yau's Theorem (Theorem 2.1.2), along with Lemma 2.3.2 in equality 5. Rearranging this gives us

$$\int_M \phi^2 \omega^n \leq C \int |\phi| \omega^n \leq C \left(\int_M \phi^2 \omega^n \right)^{1/2},$$

using Holder's inequality in the last inequality. Then this of course gives

Add to Appendix because why not.

$$\left(\int_M \phi^2 \omega^n \right)^{1/2} \leq C$$

as desired.

Appendix

Here we record a few technical results.

Lemma A.1. *If $A_{ij}(t)$ is a family of matrices then*

$$\frac{d}{dt}A^{ij}(t) = -A^{lj}A^{ki}\frac{d}{dt}A_{kl}(t).$$

Proof. Use the fact that

$$0 = \frac{d}{dt}(A^{ij}A_{jk}).$$

□

Lemma A.2. *If $A_{ij}(t)$ is a family of non-degenerate matrices then*

$$\frac{d}{dt}\log\det A_{ij}(t) = A^{ij}\frac{d}{dt}A_{ij}(t).$$

Lemma A.3. *If $A_{ij}(t)$ is a family of non-degenerate matrices then*

$$\frac{d}{dt}\det A(t) = A^{ij}\left(\frac{d}{dt}A_{ij}(t)\right)\det A_{ij}.$$

Lemma A.4. *Let (M, g) be a compact Riemannian manifold. Then*

$$\Delta_g f = \mu \textit{ is solvable} \iff \int_M \mu dV_g = 0.$$

Moreover, such a solution f is unique up to the addition of a constant.

The necessity above is easy to see.

Lemma A.5. *Recall from high school that*

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

This formula also works for $(1,1)$-forms.

Lemma A.6. *If $\alpha_1, \ldots, \alpha_n$ are non-negative $(1,1)$-forms then $\alpha_1 \wedge \cdots \wedge \alpha_n$ is positive.*

Surely this is non-negative.

Lemma A.7. *For a $(1,1)$-form $\alpha = \sqrt{-1}\alpha_{i\bar{j}}dz^i \wedge d\bar{z}^j$ then*

$$n\alpha \wedge \omega^{n-1} = \mathrm{Tr}_\omega(\alpha)\omega^n.$$