University of Waterloo

CO 739:
Several Complex Variables

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Contents

0 Introduction

This is a course on several complex variables. Prerequisites include complex analysis in a single variable and some linear algebra. Knowledge of differential forms would be useful as well.

This is a course offered through the Fields Institute, taught by Rasul Shafikov, a professor at the University of Western Ontario.

This course will be following three books broadly. They are:

- Cox-Little-O'Shea
- Ener-Herzog
- Miller-Sturmfels

Please note that these notes may have errors in them, either from my taking them down badly, or from Rasul miswriting something. Please also note the lack of pictures. This will be remedied at some point.

1 Review of Complex Analysis in one variable.

Before we start with several variables, we give a short review of single variable complex analysis.

1.1 Holomorphic Functions

Definition. We call $\Omega \subset \mathbb{C}$ a domain if Ω is open and connected.

Definition. Let $u \in C^1(\Omega)$, with $z = x + iy$, $x, y \in \mathbb{R}, z \in \mathbb{R}$. Then u is a complex function of x, y. We define the exterior derivative of u as

$$\begin{aligned} \mathrm{d}u &= \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y \qquad \text{(this is a 1-form)}, \\ &=: \frac{\partial u}{\partial z}\,\mathrm{d}z + \frac{\partial u}{\partial \bar{z}}\,\mathrm{d}\bar{z}, \end{aligned}$$

where

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right), \\ \frac{\partial u}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right), \\ \mathrm{d}z &= \mathrm{d}x + i\,\mathrm{d}y, \\ \mathrm{d}\bar{z} &= \mathrm{d}x - i\,\mathrm{d}y. \end{aligned}$$

Definition. $u \in C^1(\Omega)$ is holomorphic if $\frac{\partial u}{\partial \bar{z}} = 0$ on Ω.

Note. Playing with the definition of $\frac{\partial u}{\partial \bar{z}}$ gives the Cauchy-Riemann equations as an equivalent definition:

$$\begin{aligned} \frac{\partial \Re u}{\partial x} &= \frac{\partial \Im u}{\partial y} \\ \frac{\partial \Im u}{\partial x} &= -\frac{\partial \Re u}{\partial y}. \end{aligned}$$

Note. Differentiability of f is a property at a point, but holomorphicity is a property of f on an open set (not just a point).

Definition. Define

$$\mathcal{O}(\Omega) = \{\,\text{holomorphic functions on } \Omega\,\} \qquad \text{(also written } A(\Omega)\text{)}.$$

Example 1.1. Holomorphic functions include:

- z^n
- e^z
- Products, sums, compositions of holomorphic functions are holomorphic. Notably polynomials in z are holomorphic, and rational functions $\frac{p(z)}{q(z)}$ are holomorphic away from the zeroes of q (called poles of $\frac{p(z)}{q(z)}$).

$\triangleleft$

Non-Example 1.1. A function which is not holomorphic is $f(z) = z\bar{z} = |z|^2$. Morally this is since $\bar{z}$ is in the formula for f. $\triangleleft$

Recall. Recall that if $u : \mathbb{R}^2_{(x,y)} \to \mathbb{R}^2_{(x',y')}$ with $u = (u_1, u_2)$ then the Jacobian of u is

$$\operatorname{Jac}(u) = \det \underbrace{\begin{vmatrix} u_{1x} & u_{1y} \\ u_{2x} & u_{2y} \end{vmatrix}}_{Du}.$$

If $\operatorname{Jac} u(x_0, y_0) \neq 0$ then u is locally invertible and u^{-1} is differentiable with

$$\left(Du^{-1}\right)(u(x_0, y_0)) = \left[Du(x_0, y_0)\right]^{-1}.$$

Exercise. A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ can be thought of as a map $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$. If

$$\frac{\partial f}{\partial z}(z_0) =: f'(z_0) \neq 0$$

then prove that f is invertible and that

$$\operatorname{Jac} \tilde{f}(z_0) = \left|f'(z_0)\right|^2.$$

1.2 Cauchy Integral Formula

Probably the most important result in complex analysis in one dimension is the so-called Cauchy integral formula.

Theorem 1.1. *Let $u \in C^1(\Omega)$, Ω bounded, $\mathrm{b}\Omega$ piecewise smooth. Then $\forall z \in \Omega$*

$$u(z) = \frac{1}{2\pi i} \int_{\mathrm{b}\Omega} \frac{u(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \frac{1}{\pi} \iint_{\Omega} \frac{\partial u / \partial \bar{z}}{\zeta - z} \, \mathrm{d}\zeta \, \mathrm{d}\bar{\zeta}.$$

Note that formally $\mathrm{d}\zeta \, \mathrm{d}\bar{\zeta}$ should have a wedge product to be $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$.

Proof. This is an application of Green's formula (which itself is an application of Stokes' Theorem). More or less you realize that z is a pole of $\frac{\partial u/\partial \bar{z}}{\zeta - z}$, then define $\Omega_\epsilon = \Omega - \mathbb{B}(z, \epsilon)$ and apply Green's Formula to Ω_ϵ. $\square$

Remark. If u is holomorphic on Ω, then $\partial u / \partial \bar{z} = 0$. Then

$$u(z) = \frac{1}{2\pi i} \int_{\mathrm{b}\Omega} \frac{u(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

This is called the Cauchy Integral Formula. Note that the integrand depends holomorphically on z. Morally this says that values of u on $\mathrm{b}\Omega$ completely determines u on Ω.

Remark. $\frac{1}{\zeta - z}$ is the Cauchy kernel. This is some sense a "holomorphic reproducing kernel", since integrating a holomorphic function against it will give another holomorphic function.

Corollary 1.1. *Let $u \in \mathcal{O}(\Omega)$. Then $u \in C^\infty(\Omega)$.*

Proof. We can pull the derivative under the integral. In particular we have that $u'(z) \in \mathcal{O}(\Omega)$. □

Corollary 1.2. *Let $u \in \mathcal{O}(\Omega), z_0 \in \Omega$. Let $r > 0$ be small such that $\mathbb{B}(z_0, r) \subset \Omega$. Then in $\mathbb{B}(z_0, r)$ we have a power representation for $u(z)$:*

$$u(z) = \sum_{n \geq 0} c_n (z - z_0)^n,$$
$$c_n = \frac{1}{2\pi i} \int_{b\Omega} \frac{u(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

This means that $u \in \mathcal{O}(\Omega) \implies u$ is real analytic.

Proof. We write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z - 0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)} = \sum_{n \geq 0} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

Then we put this back into the cauchy Integral Formula. This converges since $\mathbb{B}(z_0, r) \subset \Omega$. □

Note that the converse of this also holds.

Corollary 1.3. *If $\sum_{n \geq 0} a_n (z - z_0)^n$ converges on $\mathbb{B}(z_0, r), r \geq 0$, then the the sum is holomorphic.*

In order to prove this we need the following results.

Theorem 1.2. *If $f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$ then we have that*

$$\int_{b\Omega} f(z) \, dz = 0.$$

ie $f(z)\, dz$ is a closed form (by Stokes).

Theorem 1.3. *Let $f \in C(\Omega)$ and $\int_{bR} f(z)\, dz = 0$ for all rectangles R in Ω. Then $f \in \mathcal{O}(\Omega)$.*

Note. If $F(z) := \int_{z_0}^{z} f(\zeta)\, d\zeta$, this shows that that $F \in \mathcal{O}(\Omega) \implies f \in \mathcal{O}(\Omega)$.

Corollary 1.4. *If $\{u_n\}$ are holomorphic functions on Ω which converge uniformly, that is to say that $\|u - u_n\|_k \to 0$ as $n \to \infty$, on any compact $K \subset \Omega$ to a function u, then $u \in \mathcal{O}(\Omega)$.*

Proof. Partial sums (polynomials) converge to the sum uniformly on compact $\mathbb{B}(z_0, r)$. □

Exercise. Let $u(z) \not\equiv 0$ be holomorphic near z_0, and let $u(z_0) = 0$. Show that

$$u(z) = (z - z_0)^k g(z),$$

for some $k < \infty$, where $g(z_0) \neq 0$.

Proof. Use the fact that holomorphicity is equivalent to having a power series representation. □

Theorem 1.4. *Let $\Omega \subset \mathbb{C}$ be bounded, $f \in \mathcal{O}(\Omega) \cap C(\Omega)$. Then $\max|f|$ is attained on $b\Omega$.*

Quiz Question. Let $f \in \mathcal{O}(\mathbb{C})$ be an entire function, and let $K \Subset \mathbb{C}$ be compact. Is it trye that f can be uniformly approximated on K by holomorphic polynomials?

Quiz Answer. True. Let $R > 0$ such that $\mathbb{B}(0, R) \supset K$, then take partial sums of the power series representation of f on $\mathbb{B}(0, R)$. This gives a uniform approximation on K.

1.3 Runge Theorem and Convexity

Theorem 1.5. *Let $\Omega \subset \mathbb{C}$ be bounded and $K \Subset \Omega$ be compact. TFAE:*

a) Every holomorphic function in a neighbourhood of K can be approximated uniformly on K by functions in $\mathcal{O}(\Omega)$.

b) $\Omega - K$ has no components relatively compact in Ω.

c) $\forall z \in \Omega - K$, there exists $f \in \mathcal{O}(\Omega)$ such that

$$|f(z)| > \sup_{w \in K} |f(w)|.$$

Example 1.2. Some drawings that I have not copied. ◁

Example 1.3. Let $\Omega = \mathbb{C}$, $K = \{ |z| = 1 \}$. Then $f(z) = \frac{1}{z}$ is holomorphic in a neighbourhood of K, but f cannot be approximated by entire functions, as $f \,|_K = \bar{z}$. Note that $\mathbb{C} - K$ has 2 connected components, one of which ($\{ |z| < 1 \}$) is relatively compact, so condition $b)$ fails. ◁

Proof.

$\boxed{c) \implies b)}$: Suppose that $b)$ is false, so $\exists A$ connected component of $\Omega - K$ such that $\overline{A} \Subset \Omega$. Then $\mathrm{b}A \subset K$, and by the maximum principle for holomorphic functions $\forall f \in \Lambda(\mathcal{O})$

$$\sup_{\overline{A}} |f| = \sup_{\mathrm{b}A} |f| \le \sup_K |f|,$$

but this contradicts $c)$ by taking $z \in A$.

$\boxed{a) \implies b)}$ We proceed by contradiction. Suppose there exists $A \subset (\Omega - K)$ is relatively compact. Now let $f \in \mathcal{O}(K)$, and choose $\{ f_n \} \subset \mathcal{O}(\Omega)$ such that $f_n \to f$ uniformly on K. By the maximum principle we have that

$$\sup_{\overline{A}} |f_n - f_m| = \sup_{\mathrm{b}A} |f_n - f_m| \le \sup_K |f_n - f_m| \to 0.$$

Thus $\{ f_n \}$ converges uniformly on $\overline{A}$ to a holomorphic function $F \in C(\overline{A})$.

Then pick $f(z) = \frac{1}{z - z_0}$ for some $z_0 \in A$. Then $(z - z_0)F(z) = 1$ on $\mathrm{b}A$, and so $(z - z_0)F(z) = 1$ on A. This implies that F has a pole at z_0, which is a contradiction.

$\boxed{b) \implies a)}$ To prove this we need two propositions.

Proposition 1.1. *Let $K \subset \mathbb{C}$ be compact. Then any $f \in \mathcal{O}(K)$ can be uniformly approximated on K by rational functions with poles off K.*

Proof. Let $D \supset K$ be a domain with $\mathrm{b}D$ smooth such that $f \in \mathcal{O}(\overline{D})$. By the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\mathrm{b}D} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \qquad \forall z \in D.$$

We can chop bD into a union of small arcs γ_j such that $\gamma_j \subset \mathbb{B}\left(c_j, r_j\right)$ where $c_j \in \mathrm{b}D$ and $\mathbb{B}\left(c_j, r_j\right) \cap K = \emptyset$. Then $f(z) = \sum_j f_j(z)$ where

$$f_j(z) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

Note that f_j is analytic off γ_j, as we can differentiate by $\overline{z}$ (there is no dependence on $\overline{z}$), and it vanishes at ∞.

It follows that f_j has a Laurent series expansion in descending powers of $(z - c_j)$ that converge uniformly on $\mathbb{C} - \mathbb{B}\left(c_j, r_j\right)$, especially on K. Thus $f_j(z)$ can be approximated on K by polynomials in $\frac{1}{z-c_j}$. Adding these polynomials gives the required approximation. □

Proposition 1.2. *Let $K \subset \mathbb{C}$, $U \subset \overline{\mathbb{C}} - K$ be open and connected. Let $z_0 \in U$. Then any rational function with poles in U can be approximated uniformly on K by rational functions with a pole only at z_0.*

Proof. Let

$$V = \left\{ \zeta \in U \mid \tfrac{1}{z-\zeta} \text{ can be approximated on } K \text{ by rational functions with poles at } z_0 \right\}.$$

Then $V \subset U$ and $z_0 \in V$. If we show that V is open and closed in U we have that $V = U$. Closedness is clear – let $U \ni \zeta = \lim_j \zeta_j$ for $\zeta_j \in V$. Then $\frac{1}{1-\zeta_j} \to \frac{1}{1-\zeta}$ uniformly on K.

Openness is less immediate – we need that if $\zeta_0 \in V$ and ζ is sufficiently close to z_0 then $\frac{1}{z-\zeta}$ is approximable on K by polynomials in $\frac{1}{z-\zeta_0}$. We split into cases.

Case 1: $\zeta_0 = \infty$. If $|\zeta| \gg 1$ then $\frac{1}{z-\zeta}$ is uniformly approximable on K by polynomials in z, ie rational functions with poles at ∞.
Indeed expand

$$\frac{1}{z-\zeta} = \frac{-1}{\zeta} \frac{1}{1 - \frac{z}{\zeta}} = \frac{-1}{\zeta} \sum_{k \geq 0} \frac{z^k}{\zeta^k}.$$

If $|\zeta| \gg 1$ then $|\frac{z}{\zeta}| < \frac{1}{2}$ on K. By the Weierstrass M-test, the series converges uniformly on K, and so $\zeta \in V$.

Case 2: If ζ_0 is finite then we can change coordinates on $\overline{\mathbb{C}} = \mathbb{CP}^1$ such that $z_0 \to \infty$, and then repeat the argument as in case 1.

□

To finish the proof we let $f \in \mathcal{O}(K)$, approximate by rational functions, then use the moving poles proposition to move poles of approximating sequence away from Ω.

For the remaining direction $b) \implies c)$ consult Hormander section 1.3. □

Definition. Let $K \Subset \Omega \subset \mathbb{C}$. Define the holomorphically convex hull of K in Ω to be

$$\hat{K}_\Omega := \left\{ z \in \Omega \mid \left|f(z)\right| \leq \sup_K \left|f\right|, \forall f \in \mathcal{O}(\Omega) \right\}.$$

This is quite a strange definition at first! It is easier to conceptualize with a few remarks on it.

Remark. If $z_0 \in \mathbb{C} - \Omega$ then $f(z) = \frac{1}{z-z_0} \in \mathcal{O}(\Omega)$, so $\text{dist}(K, b\Omega) = \text{dist}(\hat{K}_\Omega, b\Omega)$.

Remark. Recall that $K \subset \mathbb{R}^2$ is convex if $\mathbb{R}^2 - K$ is the union of open halfspaces. If $K \Subset \mathbb{R}^2$, the convex hull of K is the smallest convex compact set containing K.

Remark. $\hat{K}_\Omega$ is contained in the convex hull of K. Why is this so? This is since

$$|e^z| = e^{\Re(z)},$$

which gives a function, where to the left of the line $\Re(z) = 0$ the modulus is smaller than on the line, while to the right of the line the modulus is larger. Along with multiplication by constants $\alpha \in \mathbb{C}$ and shifts by a constant in $\mathbb{C}$ this gives all halfspaces.

Theorem 1.6. *Let $K \Subset \Omega$. Then*

$$\hat{K}_\Omega = K \cup \{ \textit{relatively compact components of } \Omega - K \} .$$

Proof. Consult Hormander section 1.3. Morally use the implication $b) \implies c)$ in Runge's theorem. □

Corollary 1.5. *Any domain $\Omega \subset \mathbb{C}$ can be exhausted by compact sets K_j (ie $K_1 \Subset K_2 \Subset K_3 \Subset \cdots$ such that $\bigcup_{j \geq 1} K_j = \Omega$) such that $K_j = \widehat{K_j}_\Omega$ (ie K_j is holomorphically convex).*

Definition. A function f is called meromorphic on $\Omega \subset \mathbb{C}$ if there exists a discrete set $P \subset \Omega$ such that $f \in \mathcal{O}(\Omega - P)$ and $\forall p \in P$, p is a pole for f (ie near p f can be written $f = \frac{h}{g}$ where h, g are holomorphic near p).

Remark. f meromorphic on Ω can be veiwed as a holomorphic map

$$f : \Omega \to \mathbb{CP}^1 \cong \mathbb{C} \cup \{ \infty \} .$$

Theorem 1.7. *Let $\{ z_j \}_{j=1}^\infty \subset \Omega$ be a discrete sequence of points, and let f_j be meromorphic functions near z_j. Then there exists a meromorphic function f in Ω such that f is holomorphic on $\Omega - \bigcup_{j=1}^\infty \{ z_j \}$ and $f - f_j$ is holomorphic near z_j for all j.*

Proof. We cannot naively sum the f_j's (the sum may not converge) though if there are finitely many points z_j this works. Instead we use Runge's theorem to find good approximations, in the sense that if we subtract them from the sum convergence is guaranteed, while maintaining holomorphicity of $f - f_j$ near z_j. □

Theorem 1.8. *Let $\{ z_j \}_{j=1}^\infty$ be a discrete subset of $\Omega \subset \mathbb{C}$, and let $n_j \in \mathbb{Z}$. Then there exists and meromorphic function f on Ω such that $f \in \mathcal{O}(\Omega - \bigcup_j \{ z_j \}), f \not\equiv 0$ on Ω. Moreoever $f(z) \cdot (z - z_j)^{-n_j}$ is holomorphic and $\neq 0$ near z_j for all 0. That is to say that f has prescribed 0's and poles in Ω.*

Note. If there are a finite number of z_j we can let

$$f(z) = \prod_{j < \infty} (z - z_j)^{n_j}.$$

If there are infinitely many it is less straightforward.

Remark. Mittag-Leffler prescribes poles and principal part of the Laurent expansions, while Weierstrass prescribes 0's and poles, along with degrees, but not the principal part.

Corollary 1.6. *Any meromorphic function f on Ω can be written as $f = \frac{h}{g}$ where $h, g \in \mathcal{O}(\Omega)$.*

Proof. Apply Weierstrass's theorem to poles and 0's of f separately to get respectively h, g. □

Corollary 1.7. *Given any domain $\Omega \subset \mathbb{C}$ there exists $f \in \mathcal{O}(\Omega)$ that can't be extended beyond Ω, evan as a meromorphic function.*

Proof. Let $\{ z_j \}$ be such that $\overline{\{ z_j \}}$ is dense subset of $b\Omega$. Then use Weierstrass's theorem on these to get a function with non-separable poles. □

1.4 Subharmonic Functions

Definition. $h : \mathbb{R}^2 \to \mathbb{R}^2$ is harmonic if $h \in C^2(\mathbb{R}^2)$ and

$$-\Delta h := -\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) = 0 \qquad \text{on } \mathbb{R}^2.$$

Example 1.4. For $f \in \mathcal{O}(\Omega)$, $\Re f$ and $\Im f$ are harmonic functions on Ω. Thus if h is harmonic then h is C^∞ (ie real-analytic). ◁

Theorem 1.9. *Let h be harmonic. Then*

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(z + re^{i\theta})\, \mathrm{d}\theta.$$

Definition. Let $\Omega \subset \mathbb{C}$. We say that $u : \Omega \to \mathbb{R}$ is upper semi-continuous (USC) if $\{ z \in \Omega \mid u(z) < s \}$ is open in Ω for any $s \in \mathbb{R}$.

Equivalently u is USC at $z_0 \in \Omega$ if $\forall \epsilon > 0$ there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies u(z) - u(z_0) < \epsilon.$$

Note. If we had $|u(z) - u(z_0)|$ instead of $u(z) - u(z_0)$ this would be definition of continuity. Swapping $<$ to $>$ would give the definition for lower semi-continuous.

Example 1.5. Some pictures. ◁

Definition. Let $\Omega \subset \mathbb{C}$. A function $u : \Omega \to [-\infty, \infty)$ is called subharmonic if

1. u is USC on Ω,
2. $\forall K \Subset \Omega, \forall h \in C(K)$ harmonic on K° and $h \geq u$ on bK, we have $u \leq h$ on K.

Note. In the definition above it suffices to take $K = \overline{\mathbb{B}(a, r)} \subset \Omega$ for some $a \in \Omega, r > 0$.

Note. Depending on the author, $u \equiv -\infty$ is subharmonic or not. If you do consider it to be subharmonic, you need to treat it separately most times as an edge case.

Example 1.6. For $f \in \mathcal{O}(\Omega)$ then $\log|f|$ and $|f|$ are subharmonic. ◁

Theorem 1.10. *Let $SH(\Omega)$ be the space of subharmonic functions on Ω.*

1. *For $u \in SH(\Omega)$ and $c > 0$, $cu \in SH(\Omega)$.*

2. *For $\{ u_\alpha \}_{\alpha \in A}$ are subharmonic, let $u = \sup_\alpha u_\alpha$ (pointwise). If u is USC and $u < \infty$ then u is subharmonic.*

3. *If $\{ u_j \}$ is a decreasing sequence of subharmonic functions, then $u = \lim_j u_j$ is subharmonic.*

Proof.

1. Left as an exercise.

2. Left as an exercise.

3. Let $s \in \mathbb{R}$ and consider

$$\{ z \in \Omega \mid u(z) < s \} = \bigcup_j \underbrace{\{ z \in \Omega \mid u_j(z) < s \}}_{open}.$$

Then since a union of open sets is open, it follows that u is USC. Now let h be harmonic on some $K \subset \Omega$ and suppose that $u \leq h$ on $\mathrm{b}K$. Let $\epsilon > 0$. Then $\{ z \in \mathrm{b}K \mid u_j \geq h + \epsilon \}$ is compact and decreasing in j, and so it follows that the limit is $\emptyset$. Thus $\exists j_0$ such that $u_j \leq h + \epsilon$ for all $j \geq j_0$. Since ϵ is arbitrary this implies that $u \leq h$ in K as desired.

□

Theorem 1.11. *Suppose u is subharmonic on $\Omega \subset \mathbb{C}$, and that $\max_\Omega u = u(z_0)$ for some $z_0 \in \Omega$. Then u is identically a constant.*

Proof. Suppose that $u \not\equiv$ cst. Then $\exists \zeta_1 \in \overline{\mathbb{B}(z_0, r)}$ with $u(\zeta_1) < u(z_0)$ where $r = |z_0 - \zeta_1|$.

Since u is USC $\exists$ an arc $\gamma \ni \zeta_1$ such that

$$u(\zeta) < u(z_0) - \epsilon \qquad \text{on } \gamma \text{ for some } \epsilon > 0.$$

Take $\gamma' \Subset \gamma$ and construct a function $h \in C(\mathrm{b}\mathbb{B}(z_0, r))$ such that

$$\begin{aligned} h &= u(z_0) - \epsilon && \text{on } \gamma', \\ h &= u(z_0) && \text{on } \mathrm{b}\mathbb{B}(z_0, r) - \gamma. \end{aligned}$$

Note that we can ensure that h varies linearly with respect to angle on $\gamma - \gamma'$. Now let $\tilde{h}$ be a harmonic extension of h to $\mathbb{B}(z_0, r)$. Note that $\tilde{h}$ comes from the Poisson integral of h, which gives another example of a kernel, in this case a "harmonic" one.

On $\mathrm{b}\mathbb{B}(z_0, r)$ we have $u \leq h$ by construction, and

$$u(z_0) \leq \tilde{h}(z_0) = \frac{1}{2\pi} \int_{\mathrm{b}\mathbb{B}(z_0,r)} h(\zeta)\, \mathrm{d}\zeta < u(z_0),$$

where the last inequality comes from the fact that the LHS is the average of h on $\mathrm{b}\mathbb{B}(z_0, r)$. This yields a contradiction, thus u is identically constant. □

Theorem 1.12. *Let u be USC on Ω. Then*

$$u \text{ is subharmonic} \iff u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, \mathrm{d}\theta \quad \forall z \in \Omega, r > 0 \text{ st. } \mathbb{B}(z_0, r) \subset \Omega.$$

Proof. Approximate u with harmonic functions on the boundary and apply the MVT for harmonic functions. □

Corollary 1.8. *If $f \in \mathcal{O}(\Omega)$ then $u = \ln|f|$ and $u = |f|$ are subharmonic.*

Proof. Apply MVT and max modulus bound. ☐

Remark.

- Let u be subharmonic. Then $\{ z \mid u(z) = -\infty \}$ is called the polar set of u. It is known that this set has Lebesgue measure 0 (or is all of $\mathbb{C}$ in the edge case that $u \equiv -\infty$).
- If $u \in C^2(\Omega)$ then

$$u \text{ subharmonic} \iff -\Delta u \leq 0.$$

- Subharmonic functions can be defined on $\mathbb{R}^n$ for $n \geq 1$.

2 Basic Complex Analysis in Several Variables

2.1 Holomorphic Functions on $\mathbb{C}^n$

First we will talk about the geometry of $\mathbb{C}^n$.

Remark.

- We define $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$.
- $z \in \mathbb{C}^n \implies z = (z_1, \cdots, z_n)$ with $z_j \in \mathbb{C}$.
- There is a natural Hermitian inner product on $\mathbb{C}^n$ given by

$$(a, b) = \sum_{j=1}^{n} a_j \overline{b_j}.$$

- This induces the norm

$$|a| = \sqrt{(a, a)}.$$

- It is true that $\Re(a, b)$ is the euclidean dot product on $\mathbb{C}^n \cong \mathbb{R}^{2n}$.
- It is true that $\Im(a, b)$ is the "standard symplectic form" on $\mathbb{C}^n$.
- We can compactify $\mathbb{C}$ to get $\mathbb{C} \cup \{ \infty \} = \mathbb{CP}^1$ (also called $\mathbb{P}^1$ in complex algebraic geometry). This is the Riemann sphere and the most classical example of an interesting Riemann surface. It is possible to compactify $\mathbb{C}^n$ as well. Thinking geometrically, we must add one point at infinity for each complex line (ie a copy of $\mathbb{C}$) going through the origin. This is exactly the Grassmannian

$$Gr(n, 1) \cong \mathbb{CP}^{n-1}.$$

  Then we have

$$\text{compactification of } \mathbb{C}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1} = \mathbb{CP}^n.$$

  This is a complex manifold of dimension n, ie a real manifold of dimension $2n$.

Definition. There are a few important examples of domains in $\mathbb{C}^n$.

- The unit ball $\mathbb{B}^n = \{ z \in \mathbb{C}^n \mid |z| < 1 \}$.
- The unit polydisc $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$, where $\mathbb{D} = \mathbb{B}^1$.
- The ball $\mathbb{B}^n(z_0, r) = \{ z \in \mathbb{C}^n \mid |z - z_0| < r \}$.
- $\mathbb{D}^n(z,r) = \mathbb{D}(z_1, r_1) \times \cdots \times \mathbb{D}(z_n, r_n)$, where $z = (z_1, \cdots, z_n)$ and $r = (r_1, \cdots, r_n)$ is a polyradius.

Definition. The absolute mapping of a set is a function from $\mathbb{C}^n$ to so-called absolute space $\mathbb{R}^n_+$ given by

$$\tau(z_1, \cdots, z_n) = (|z_1|, \cdots, |z_n|).$$

Example 2.1. absolute space of a ball and a polydisc. ◁

Definition. Let $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ and $r = (r', r_n) \in \mathbb{R}^n_+$ where $0 < |r_j| < 1$ for all j. We define a Hartogs domain as

$$\begin{aligned} H(r) = &\{ z \in \mathbb{C}^n \mid z' \in \mathbb{D}^{n-1}(0, r'), |z_n| < 1 \} \\ &\cup \{ z \in \mathbb{C}^n \mid z' \in \mathbb{D}^{n-1}(0,1), r_n < |z_n| < 1 \}. \end{aligned}$$

This is a domain as the union of two open sets. This is also realizable as a polydisc with another polydisc removed.

Remark. Let $r = (r_1, \cdots, r_n) \in \mathbb{R}^n_+$ be a point. Then

$$\begin{aligned} \tau^{-1}(r) &= \{ (r_1 e^{i\theta_1}, \cdots, r_n e^{i\theta_n}) \mid 0 \le \theta_j \le 2\pi \} \\ &\cong S^1 \times \cdots \times S^1 \\ &= n\text{-torus}. \end{aligned}$$

Definition. $\Omega \subset \mathbb{C}^n$ is called a Reinhardt domain (around 0) if $\forall a \in \Omega$

$$\tau^{-1}\left(\tau(a)\right) = \{ (a_1 e^{i\theta_1}, \cdots, a_n e^{i\theta_n}) \} \subset \Omega.$$

Ω is a complete Reinhardt domain if

$$a \in \Omega \implies \mathbb{D}^n(0, \tau(a)) \subset \Omega.$$

Example 2.2. both ball and polydisc are CRDs. $H(r)$ is an RD but not a CRD. ◁

Note. We use standard multi-index notation. Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$. Then define

$$\begin{aligned} |\alpha| &= \alpha_1 + \cdots + \alpha_n \\ \alpha! &= \alpha_1! \cdots \alpha_n! \\ D^\alpha &= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \qquad \text{on } \mathbb{R}^n_{(x_1, \cdots, x_n)}. \end{aligned}$$

Then we can define

$$\begin{aligned} \frac{\partial}{\partial z_j} &:= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \\ \frac{\partial}{\partial \overline{z_j}} &:= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right). \end{aligned}$$

Exercise. Show that

$$\overline{\left(\frac{\partial f}{\partial z_j}\right)} = \frac{\partial \overline{f}}{\partial \overline{z_j}}$$

and

$$\overline{\left(\frac{\partial f}{\partial \overline{z_j}}\right)} = \frac{\partial \overline{f}}{\partial z_j}.$$

Note. One should not think of Wirtinger derivatives as directional derivatives, but as linear combinations of derivatives.

Note. We use the notation

$$D^{\alpha\overline{\beta}} = \frac{\partial^{|\alpha|+|\beta|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n} \partial \overline{z_1}^{\beta_1} \cdots \partial \overline{z_n}^{\beta_n}},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$.

Definition. Let $D \subset \mathbb{C}^n$ be a domain. hen we say $f : D \to \mathbb{C}$ is holomorphic if $f \in C^1(D)$ and

$$\frac{\partial f}{\partial \overline{z_j}}(z) = 0$$

for all $j \in \{1, \cdots, n\}$ and $z \in D$. We write the set of holomorphic functions on D as $\mathcal{O}(D)$.

Note. $f \in C^1(D) \implies$ all first order partial derivatives $\frac{\partial f}{\partial x_j}$ and $\frac{\partial f}{\partial y_j}$ are continuous on D. This is equivalent to being able to write

$$f(z) = f(z_0) + \mathrm{d}f(z_0)(z - z_0) + o(|z - z_0|)$$

for z close to z_0.

Example 2.3. Polynomials in z are holomorphic. Sums, differences, and products of holomorphic functions are holomorphic. Fractions of holomorphic functions $\frac{f}{g}$ are holomorphic if $g \neq 0$. ◁

Definition. We define the differential of f at $a \in D$ to be

$$\begin{aligned} \mathrm{d}f_a &= \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a)\,\mathrm{d}x_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}(a)\,\mathrm{d}y_j}_{\text{linear map } \mathbb{R}^{2n} \to \mathbb{R}} \\ &= \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a)\,\mathrm{d}z_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z_j}}(a)\,\mathrm{d}\overline{z_j}}_{\text{linear map } \mathbb{C}^{n} \to \mathbb{C}} \end{aligned}$$

Remark. A linear map can be $\mathbb{R}$-linear (real-linear) or $\mathbb{C}$-linear (complex-linear). A linear map $A : \mathbb{R}^{2n} \to \mathbb{R}^2$ corresponds to a linear space of dimensions $\dim_{\mathbb{R}} = 2n$ in $\mathbb{R}^{2n} \times \mathbb{R}^n$ (the graph of A). A map is complex-linear if this $2n$-dimension real space is in fact a complex subspace of $\mathbb{C}^n \times \mathbb{C}$.

Example 2.4. On $\mathbb{C}^n$, let $z_j = x_j + iy_j$ for $j = 1, \cdots, n$. Then

$$\mathbb{C}^{n-1} = \operatorname{span}_{\mathbb{C}} \{ z_1, \cdots, z_{d-1} \}$$

is complex linear, but

$$\operatorname{span}_{\mathbb{R}} \{ z_1, \cdots, z_{n-2}, x_{n-1}, x_n \}$$

is real but not complex linear. $\triangleleft$

Remark. From a differential perspective a C^1-smooth function $f : D \to \mathbb{C}$ is holomorphic if $\mathrm{d}f_a$ is $\mathbb{C}$-linear for any $a \in D$.

Picture of function and differential

Here then $T_{(a,f(a))}\Gamma_f =$ graph of $\mathrm{d}f_a \cong$ lin. subspace of $\mathbb{C}^n \times \mathbb{C}$. So the space $T_{(a,f(a))}\Gamma_f$ viewed as a subspace of $\mathbb{C}^n \times \mathbb{C}$ is a complex linear subspace. If this is true $\forall a \in D$, Γ_f is the graph of a holomorphic function, ie f is holomorphic. Then

$$\text{CR-equations: } \left.\frac{\partial f}{\partial \overline{z_j}}\right|_{z=a} = 0 \quad \forall a \in D \iff \mathrm{d}f_a \text{ is } \mathbb{C}\text{-linear at every } a \in D.$$

2.2 Cauchy Integral Formula

The Cauchy integral formula is not as fundamental as in a single variable, but it is still useful.

Theorem 2.1. *Let $\mathbb{D}^n(a,r) = \mathbb{D}(a_1, r_1) \times \cdots, \mathbb{D}(a_n, r_n)$ with $\mathbb{D}(a_j, r_j) \subset \mathbb{C}_{z_j}$. Suppose $f \in C(\overline{\mathbb{D}^n(a,r)})$ and f is holomorphic in each variable z_j, ie*

$$\tilde{f}(\lambda) = f(z_1, \cdots, z_{j-1}, \lambda, z_{j+1}, \cdots, z_d) : \mathbb{D}(a_j, r_j) \to \mathbb{C}$$

is holomorphic in λ. Then $\forall z \in \mathbb{D}^n(a,r)$

$$f(z) = \frac{1}{(2\pi i)^n} \int f(\zeta) \frac{\mathrm{d}\zeta}{(\zeta - z)} := \frac{1}{(2\pi i)^n} \int_{\mathrm{b}_0\mathbb{D}^n(a,r)} f(\zeta_1, \cdots, \zeta_n) \frac{\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

where the integral is taken over the distinguished boundary of $\mathbb{D}^n$

$$\mathrm{b}_0\mathbb{D}^n(a,r) = \{ \zeta \in \mathbb{C}^n \mid |\, \zeta_j - a_j| = r_j, j = 1, \cdots, n \} \cong S^1 \times \cdots \times S^1.$$

Remark. $\mathrm{b}_0\mathbb{D}^n(a,r) \subsetneq \mathrm{b}\mathbb{D}^n(a,r)$, since $\mathbb{D}^n(a,r)$ has real dimension $2n$, so the boundary has real dimension $2n-2$, but $(S^1)^n$ has real dimension n.

Picture of distinguished boundary.

Remark. Consider the function $f(x,y) = \frac{xy}{x^2+y^2}$ on $\mathbb{R}^2$. This has all partial derivatives but is not continuous at the origin. Thus having partial derivatives is not a priori the same as being holomorphic.

Remark. Note that $\mathrm{b}_0\mathbb{D}^n(a,r)$ can be parameterized by $z_j = a_j + r_j e^{i\theta_j}$, with $0 \le \theta_j \le 2\pi$ for $j \in \{ 1, \cdots, n \}$. This means that

$$\int_{\mathrm{b}_0\mathbb{D}^n} g(\zeta)\, \mathrm{d}\zeta = i^n \int_{[0,2\pi]^n} g(\zeta(\theta)) e^{i\theta_1} \cdots e^{i\theta_n}\, \mathrm{d}\theta.$$

Proof. We use induction on n. If $n = 1$ this is just the standard Cauchy integral formula. Suppose now that the CIF holds for $n-1$ variables. Let $z \in \mathbb{D}^n(a, r)$. Then

$$f(z_1, \cdots, z_n) = \frac{1}{(2\pi i)^n} \int_{b_0\mathbb{D}^{n-1}(a', r')} f(z_1, \zeta_2, \cdots, \zeta_n) \frac{d\zeta_2 \cdots d\zeta_n}{(\zeta_2 - z_2) \cdots (\zeta_n - z_n)}. \tag{2.1}$$

Fix $\zeta_2, \cdots, \zeta_n$. Then

$$f(z_1, \zeta_2, \cdots, \zeta_n) = \frac{1}{2\pi i} \int_{|\zeta_1 - a_1| = r_1} f(\zeta_1, \zeta_2, \cdots, \zeta_n) \frac{d\zeta_1}{(\zeta_1 - z_1)}. \tag{2.2}$$

Plugging (2.2) into (2.1) gives us an iterated integral. Since $f \in C(\overline{\mathbb{D}^n})$ we can rearrange integrals to get the CIF. □

Corollary 2.1. *If f is continuous and holomorphic in each variable then f is holomorphic.*

Remark. By a theorem of Hartogs continuity is not required.

Corollary 2.2. *Let $f \in \mathcal{O}(\mathbb{D}^n(a, r))$. Then*

$$\left|D^\alpha f(a)\right| \leq \frac{\alpha!}{r^\alpha} \sup_{z \in \mathbb{D}^n(a,r)} \left|f(z)\right|.$$

Proof. Take a smaller r, then differentiate α times. Then write

$$(D^\alpha f)(a) = \frac{\alpha!}{(2\pi i)^n} \int_{b_0\mathbb{D}^n(a,r)} f(\zeta) \frac{d\zeta}{(z-a)^{\alpha+1}}$$

and use max modulus bound. □

2.3 Holomorphic Functions as Power Series

We will try to show that holomorphic functions have valid power series expansions.

Definition. A sequence $f_j \to f$ converges compactly (or normally) on a domain $D \subset \mathbb{C}^n$ if $\{ f_j \}$ converges uniformly on each compact subset of D.

Theorem 2.2. *Let $\{ f_j \} \subset \mathcal{O}(D)$ be a sequence such that $\lim_{j\to\infty} f_j = f$ compactly. Then $f \in \mathcal{O}(D)$ and $D^\alpha f_j \to D^\alpha f_j$ compactly.*

Remark. Compact convergence defines a topology on $\mathcal{O}(D)$. In fact $\mathcal{O}(D)$ is metrizable. More explicitly let $\{ K_j \}$ be a normal exhaustion on D, that is to say a sequence with $K_j \Subset K_{j+1}$ and $\bigcup_j K_j = D$. Then let

$$\operatorname{dist}(f, g) = \sum_{\nu=1}^{\infty} 2^{-\nu} \frac{\left|f-g\right|_{K_\nu}}{1 + \left|f-g\right|_{K_\nu}} \leq 1,$$

where $\left|f-g\right|_{K_\nu}$ is the sup-norm. This defines a valid metric, and thus defines a topology.

2.4 Power Series

Definition. A sum $\sum_{\nu\in\mathbb{N}^n} b_\nu$ is absolutely convergent if

$$\sum_{\nu\in\mathbb{N}^n} |b_\nu| = \sup\{\sum_{\nu\in\Lambda} b_\nu \mid \Lambda \text{ finite set}\} < \infty.$$

This is equivalent to $\forall$ bijections $\sigma : \mathbb{N} \to \mathbb{N}^n$ then series $\sum_{j=1}^{\infty} b_{\sigma(j)}$ converges absolutely and the limit is independent of the choice of σ.

Definition. The power series centered at $a \in \mathbb{C}^n$ is

$$\sum_{\nu\in\mathbb{N}^n} b_\nu$$

where

$$b_\nu = c_\nu(z-a)^\nu,$$

$$c_\nu = c_{\nu_1,\cdots,\nu_n} \in \mathbb{C},$$
$$(z-a)^\nu = (z_1-a_1)^{\nu_1}\cdots(z_n-a_n)^{\nu_n}.$$

We usually stick with $a = 0$ for simplicity.

Definition. The domain of convergence of a power series $\sum_{\nu\in\mathbb{N}^n} c_\nu(z-a)^\nu$ is the interior of the set of points of convergence of the series.

Note. The domain of convergence may be empty.

Example 2.5. Consider $\sum_{\nu_1,\nu_2\geq 0} \nu_1!\nu_2!z_1^{\nu_1}z_2^{\nu_2}$. This converges if $z_1z_2 = 0$ and diverges otherwise. Thus the domain of convergence is $\emptyset$. ◁

Lemma 2.1. *Let $c_\nu \in \mathbb{C}$ and $\nu \in \mathbb{N}^n$. Suppose that for some $w \in \mathbb{C}^n$ we have*

$$\sup_{\nu\in\mathbb{N}^n} |c_\nu w^\nu| < M < \infty. \tag{2.3}$$

Let $r = \tau(w) = (|w_1|,\cdots,|w_n|)$. Then $\sum c_\nu z^\nu$ converges on $\mathbb{D}^n(0,r)$. This convergence is normal, ie if $K \Subset \mathbb{D}^n(0,r)$ there exists a finite set Λ (a "tail") such that

$$\sum_{\nu\notin\Lambda} |c_\nu z^\nu| < \epsilon \qquad \forall z \in K.$$

Proof. For all $K \Subset \mathbb{D}^n(0,r)$, choose $0 < \lambda < 1$ such that $K \subset \mathbb{D}^n(0,\lambda r) \Subset \mathbb{D}^n(0,r)$. Then for $z \in \mathbb{D}^n(0,\lambda r)$ and $\forall\nu$ we have

$$|c_\nu z^\nu| \leq |c_\nu w^\nu| \cdot \lambda^{|\nu|} < M \cdot \lambda^{|\nu|}.$$

Since $\sum_{\nu\in\mathbb{N}^n} \lambda^{|\nu|}$ converges absolutely then the result holds. □

Corollary 2.3. *If a domain of convergence of a power series is non-$\emptyset$ it is a complete Reinhardt domain. It is also the interior of the set of points w that satisfy (2.3). The convergence is normal.*

Picture of domains.

Remark. Not every complete Reinhardt domain is a domain of convergence of some power series of a holomorphic function.

Definition. Let $\tau_1 : w \mapsto (\ln |w_1|, \cdots, \ln |w_n|)$. $\Omega \subset \mathbb{C}^n$ is log-convex if $\tau_1(\Omega)$ is convex in $\mathbb{R}^n$.

Proposition 2.1. *Ω is the domain of convergence of some power series $\iff$ Ω is log-convex.*

Example 2.6. A complete Reinhardt domain that is not log-convex. ◁

Theorem 2.3. *A power series $f(z) = \sum_\nu c_\nu z^\nu$ with a non-empty domain of convergence Ω defines a holomorphic function $f \in \mathcal{O}(\Omega)$. Moreover $\forall \alpha \in \mathbb{N}^n$ the series of derivatives $\sum_\nu c_\nu (D^\alpha z^\nu)$ converges compactly to $D^\alpha d$ on Ω and*

$$(D^\alpha f)(0) = \alpha! c_\alpha \tag{2.4}$$

Proof. Fix a bijection $\sigma : \mathbb{N} \to \mathbb{N}^n$. Then

$$f(z) = \lim_{k\to\infty} \sum_{j=1}^{k} c_{\sigma(j)} z^{\sigma(j)}.$$

This converges compactly on Ω. Since partial sums are polynomials and are thus holomorphic on Ω, f is holomorphic and

$$D^\alpha f(z) = \lim_{k\to\infty} \sum_{j=1}^{k} c_{\sigma(j)} (D^\alpha z^{\sigma(j)})$$

on Ω. So for a fixed $\alpha \in \mathbb{N}^n$ and $w \in \Omega$ we have

$$\sup_\nu \left| c_\nu (D^\alpha z^\nu) \right| \Big|_{z=w} < \infty.$$

Ω is contained in the domain of convergence of the power series $\sum_\nu c_\nu (D^\alpha z^\nu)$ by Lemma (2.1). To prove (2.4) one can evaluate $D^\alpha f(z)$ at $z = 0$. □

We have proven that a power series gives a holomorphic function. We now want to prove the inverse.

2.5 Taylor Series

Theorem 2.4. *Let $f \in \mathcal{O}(\mathbb{D}^n(a, r))$. Then the Taylor series of f converges to f in the polydisc, ie*

$$f(z) = \sum_{\nu\in\mathbb{N}^n} \frac{D^\nu f(a)}{\nu!} (z-a)^\nu \qquad \forall z \in \mathbb{D}^n(a, r). \tag{2.5}$$

Proof. Apply the CIF on $\mathbb{D}^n(a, \rho)$ with $\rho < r$. Then use the expansion

$$\frac{1}{\zeta - z} = \sum_{\nu\in\mathbb{N}^n} \frac{(z-a)^\nu}{(\zeta - a)^{\nu+1}}. \tag{2.6}$$

This sum converges uniformly for $\zeta \in b_0\mathbb{D}^n(a, \rho)$ because

$$\frac{\left|z_j - a_j\right|}{\left|\zeta_j - a_j\right|} \le \frac{\left|z_j - a_j\right|}{\rho} < 1.$$

Then put (2.5) into the CIF and replace the factor of $\frac{1}{\zeta - z}$ with (2.6). Since this is uniformly convergent we can swap the integral and the sum to get

$$f(z) = \sum_{\nu \in \mathbb{N}^n} \underbrace{\left[\frac{1}{(2\pi i)^n} \int_{\mathrm{b}_0 \mathbb{D}^n(a,\rho)} f(\zeta) \frac{\mathrm{d}\zeta}{(\zeta - a)^{\nu+1}} \right]}_{= \frac{D^\alpha f(a)}{\nu!} \text{ by a corollary to CIF}} (z-a)^\nu.$$

□

Remark. If $n = 1$ then $f \in \mathcal{O}(\Omega)$ with $f \not\equiv cst$ implies that $f^{-1}(0)$ is discrete. If $n > 1$ this is not so, Consider $f(z_1, z_2) = z_1$. Then $f^{-1}(0) = \{ z_1 = 0 \}$ is not discrete.

2.6 Hartogs' Theorem

There were classically two definitions for holomorphic functions.

Definition (Riemann)**.** $f \in \mathcal{O}(\Omega) \iff \frac{\partial f}{\partial \bar{z}_j} \equiv 0$ for all $j = 1, \cdots, n$.

Definition (Weierstrass)**.** $f \in \mathcal{O}(\Omega) \iff f$ locally admits a power series representation $f(z) = \sum_{\nu \in \mathbb{N}^n} c_\nu (z-a)^\nu$.

Note. A priori Riemann's definition seems unsatisfactory compared to Weierstrass's. The example is $f(x,y) = \frac{xy}{x^2+y^2}$, which has partial derivatives but is not continuous at the origin. It seems that we should require additionally $f \in C^1(\Omega)$. Hartogs proved that $f \in C^1(\Omega)$ is not necessary.

Theorem 2.5 (Hartogs, 1907)**.** *If f is holomorphic in each variable then f is holomorphic.*

To prove this we need a few lemmas. This is the first serious result in the class.

Lemma 2.2 (Schwarz)**.** *Let $\phi \in \mathcal{O}(\mathbb{D}^1(0,r))$ with $\phi(z_0) = 0$ for some $z_0 \in \mathbb{D}^1(0,r)$, and $|\phi| \leq M$ on $\mathbb{D}^1(0,r)$. Then*

$$|\phi(z)| \leq M \cdot r \frac{|z - z_0|}{|r^2 - \overline{z_0} z|}.$$

Proof. Take $\lambda : \mathbb{D}(0,r) \to \mathbb{D}(0,1) \subset \mathbb{C}$ be a conformal map such that $\lambda(z_0) = 0$ and apply the standard version of the Schwarz lemma on $\mathbb{D}(0,1)$. □

Lemma 2.3. *If f is holomorphic in each variables z_ν in $\mathbb{D}^n(a,r) = U$ and f is bounded in U then it is continuous there.*

Proof. Take any $z', z \in U$. Then one can write

$$f(z) - f(z') = \sum_{\nu=1}^{n} \left[f(z'_1, \cdots, z'_{\nu-1}, z_\nu, \cdots, z_n) - f(z'_1, \cdots, z'_\nu, z_{\nu+1}, \cdots, z_n) \right].$$

Consider each term as a function $\phi_\nu(z_\nu)$ while other variables are fixed. Say $|f| < \frac{M}{2}$ in U. Then ϕ_ν satisfies the Schwarz lemma, and so

$$|f(z) - f(z')| \leq \sum_{\nu=1}^{n} |\phi_\nu(z_\nu)| \leq M \sum_\nu r_\nu \frac{|z_\nu - z'_\nu|}{|r_\nu^2 - \overline{z_\nu} z_\nu|}.$$

Then as $z \to z_0$, $f(z) \to f(z_0)$ and so f is continuous. □

Note. If f is holomorphic in each variable and bounded, then by the previous lemma f is continuous and so by CIF f is holomorphic.

Theorem 2.6 (Baire). *Let X be a complete metric space with $X = \bigcup_{m=1}^{\infty} A_m$, where A_m is closed. Then at least one of A_m contains a nonempty open set.*

Lemma 2.4 (Osgood). *Let $U = \mathbb{D}^n(0, R)$, $z = (z', z_n)$ with $z' = (z_1, \cdots, z_{n-1})$. Write*

$$U = U' \times U_n,$$

$$\begin{aligned} U' &= \mathbb{D}^{n-1}(0', R') \subset \mathbb{C}^{n-1}_{z'}, \\ U_n &= \mathbb{D}(0, R_n) \subset \mathbb{C}_{z_n}. \end{aligned}$$

If $f(z', z_n)$ is continuous in z' on $\overline{U'}$ for any fixed $z_n \in \overline{U_n}$ and continuous in z_n on $\overline{U_n}$ for any fixed $z' \in U'$, then there exists a polydisc

$$W = W' \times U_n \subset U$$

where f is bounded.

Picture of domains.

Proof. For a fixed $z' \in U'$ write

$$M(z') = \max_{z_n \in \overline{U_n}} \left| f(z', z_n) \right|$$

and for $m \in \mathbb{N}$

$$E_m = \{ z' \in \overline{U'} \mid M(z') \leq m \}.$$

Then E_m is closed. This is as: let $\{ z'_\mu \} \subset E_m$ such that $z'_\mu \to z'_0 \in U'$. Then we want to show that $z'_0 \in E_m$. Indeed suppose $|f(z'_\mu, z_n)| \leq m$. Then by continuity of f in z' we have that $|f(z', z_n)| \leq m$ for any $z_n \in \overline{U_n}$. in other words $M(z') \leq m$, so E_m is closed.

Clearly $E_m \subseteq E_{m+1}$ and $\bigcup E_m = \overline{U'}$. we claim that there exists some $M > 0$ such that E_M contains a non-$\emptyset$ domain $G' \subset U'$. This follows from the Baire category theorem. Then $W \subset G' \times U_n$ polydisc. Then on W we have $\left| f \right| < M$. □

Note. This almost gives us our result! We know f is holomorphic in a subset, but not everywhere.

Lemma 2.5 (Hartogs). *Let $r < R$. Then define*

$$\begin{aligned} V' &= \mathbb{D}^{n-1}(a', R) \subset \mathbb{C}^{n-1}, \\ W' &= \mathbb{D}^{n-1}(a', r) \subset \mathbb{C}^{n-1}, \\ U_n &= \mathbb{D}(0, R), \end{aligned}$$

$$\begin{aligned} V &= V' \times U_n, \\ W &= W' \times U_n. \end{aligned}$$

If $f(z', z_n)$ is holomorphic in z' in V' $\forall z \in U_n$ and is holomorphic in z in $\overline{W}$, then it is holomorphic in $\overline{V}$.

Lemma 2.6. *If $\{ u_k \}$ is a sequence of subharmonic functions on $D \subset \mathbb{C}$ that are uniformly bounded on each compact subset of D, and $\forall z \in D$*

$$\limsup_{k\to\infty} u_k(z) \leq A \tag{2.7}$$

then for any compact $K \Subset D$ for all $\epsilon > 0$, there exists k_0 such that

$$u_j(z) \leq A + \epsilon$$

for all $z \in K, \forall j > k_0$.

Proof. Consult Hormander chapter 1. □

Proof of Lemma 2.5. Wlog we have that $a' = 0'$ and that for any $z_n \in U_n, z' \in U'$,

$$f(z) = \sum_{|k|=0}^{\infty} c_k(z_n)(z')^k, \qquad k = (k_1, \cdots, k_{n-1}),$$

where

$$c_k(z_n) = \frac{1}{k!}\frac{\partial^{|k|} f}{(\partial z')^k}(0', z_n).$$

These are holomorphic functions in z_n since we are varying in $\{0\} \times U_n \subset W$. This implies that $\frac{1}{|k|}\ln|c_k(z_n)|$ is subharmonic for all k.

Let $\rho < R$. Since $f(z)$ converges in V we have that $\forall z_n \in U_n$

$$|c_k(z_n)|\,\rho^{|k|} \to \infty$$

as $|k| \to \infty$. It follows that for a fixed $z_n \in U_n$ there exists $k_0 > 0$ such that $\forall |k| > k_0$ we have

$$\frac{1}{|k|}\ln|c_k(z_n)| + \ln\rho \geq 0$$

and so

$$\limsup_{|k|\to\infty} \frac{1}{|k|}\ln|c_k(z_n)| \leq \ln\frac{1}{\rho}. \tag{2.8}$$

Note that $f \in \mathcal{O}(\overline{W}) \implies f$ is bounded on $\overline{W}$, say that $|f| < M$. By the Cauchy inequality (ie that $|c_k(z_n)| \leq Mr^{-|k|}$) then

$$|c_k(z_n)|\,r^{|k|} \leq M \qquad \forall z_n \in U_n.$$

It follows that

$$\frac{1}{|k|}\ln|c_k(z_n)| \leq \ln\left(\frac{M^{1/|k|}}{r}\right) \leq M_0 \in \mathbb{R}_+. \tag{2.9}$$

Since 2.8 (limsup condition) and 2.9 (uniformly bounded condition) hold then we can apply Lemma 2.6. Then $\forall \sigma < \rho$ there exists k_0 such that $\forall |k| > k_0$ and $\forall z$ such that $|z| < \sigma$ we have that

$$\frac{1}{|k|}\ln|c_k(z_n)| \leq \ln\frac{1}{\sigma}$$

ie

$$\left|c_k(z_n)\right| \sigma^{|k|} \leq 1.$$

By Lemma 2.1 $\sum c_k(z_n)(z')^k$ converges uniformly in any $\mathbb{D}'(0,\sigma')$ where $\sigma' < \sigma$. The coefficient are continuous functions, so f is bounded and continuous. It follows that f is holomorphic. □

With our preparatory lemmas out of the way we can prove Hartogs theorem.

Proof of Theorem 2.5. Notice that this is local. Let $z^0 = 0$ be a an arbitrary point in Ω. f is holomorphic in each variable on $\mathbb{D}^n(0,R)$. We need to show that f is holomorphic in some neighbourhood of 0. We proceed by induction on the number of variables. If $n = 1$ this is trivially true. Suppose that f is holomorphic in $z' = (z_1, \cdots, z_{n-1})$.

Let $U' = \mathbb{D}^{n-1}(0', \frac{R}{3})$ and $U_n = \mathbb{D}(0,R)$. By assumption f is continuous in z' on $\overline{U'}$ for all $z_n \in \overline{U_n}$ and in $z_n \in \overline{U_n}$ for all $z' \in \overline{U'}$.

Insert image.

f is bounded in some polydisc $W = W' \times \overline{U_n}$ where $W' = \mathbb{D}^{n-1}(a', r)$. Then f is holomorphic in W.

Now consider $V' = \mathbb{D}^{n-1}(a', \frac{2R}{3})$ and $V = V' \times U_n$. Then $\overline{V} \subset \overline{\mathbb{D}(0,R)}$. f is holomorphic in z' in $\overline{V'}$ for all z_n, and holomorphic in z in $\overline{W}$. By Hartogs' lemma f is holomoprhic in $\overline{V}$. Since $0 \in V$ then f is holomorphic near 0. Then f is holomorphic at each point. □

This marks the end of basic complex analysis in several variables.

3 Holomorphic Mappings

We need a notion for holomorphicity of maps $\mathbb{C}^n \to \mathbb{C}^m$.

Definition. A map $F : \mathbb{C}^n \to \mathbb{C}^m$ is **holomorphic** if $F = (f_1, \cdots, f_m)$ with $f_j \in \mathcal{O}(\mathbb{C}^n)$. The **differential** (or **Jacobian**) of F at $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ is

$$DF(z) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_n} \end{bmatrix} (z).$$

Note this is an $n \times m$ matrix with complex coefficients. This is a "complex differential", but we can write this as a "real differential"

$$D_{\mathbb{R}}F = \begin{bmatrix} \frac{\partial \Re f_j}{\partial x_k} & \frac{\partial \Im f_j}{\partial x_k} \\ \frac{\partial \Re f_j}{\partial y_k} & \frac{\partial \Im f_j}{\partial y_k} \end{bmatrix},$$

which is a $2n \times 2m$ matrix with real coefficients.

Lemma 3.1. *We have that*

$$\det D_{\mathbb{R}}F(z) = \left|\det DF(z)\right|^2 \geq 0.$$

Remark.

- The chain rule has a natural analogue for holomorphic maps.
- $\mathbb{C}^n$ has a natural orientation. If $F : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic and $\det DF \neq 0$, then F is orientation-preserving.

3.1 Implicit Function Theorem

Theorem 3.1. *Let $D \subset \mathbb{C}^n$ be a domain, $F = (f_1, \cdots, f_m) : D \to \mathbb{C}^m$ be holomorphic with $m < n$ and $F(a) = 0$ for some $a \in D$. Further write*

$$DF(a) = \left[\begin{array}{ccc|ccc|} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_{n-m}} & \frac{\partial f_1}{\partial z_{n-m+1}} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_{n-m}} & \frac{\partial f_m}{\partial z_{n-m+1}} & \cdots & \frac{\partial f_m}{\partial z_n} \end{array} \right] (a)$$

and assume that the boxed $m \times m$ minor of DF has full rank. That is assume that

$$\det \left[\frac{\partial f_k}{\partial z_j}(a) \right]_{\substack{k=1,\cdots,m \\ j=n-m+1,\cdots,n}} \neq 0.$$

Then there exists a holomorphic map

$$h = (h_1, \cdots, h_m) : \mathbb{B}'(a', \epsilon') \to \mathbb{B}''(a'', \epsilon'')$$

where $a' \in \mathbb{C}^n, a'' \in \mathbb{C}^m$, and $a = (a', a'')$ such that

$$F(z', z'') = 0 \iff z'' = h(z').$$

Picture of Implicit function theorem

3.2 Biholomorphic Maps

Theorem 3.2 (Inverse Function Theorem)**.** *Let $D \subset \mathbb{C}^n, F : D \to \mathbb{C}^n$ with $\det DF(a) \neq 0$. Then there exist neighbourhoods U of a and W of $b = f(a)$ such that*

$$F|_U : U \to W$$

*is a homeomorphism and $F^{-1} : W \to U$. In this case then $F|_U$ is called **biholomorphic**).*

Proof. Let $w \in \mathbb{C}^n$. Define $G(w, z) = F(z) - w$. This is a holomorphic function from $\mathbb{C}^n \times D \to \mathbb{C}^n$. Note that $G(a, b) = 0$. Say that $G = (g_1, \cdots, g_n)$. Then

$$\det \left[\frac{\partial g_k}{\partial z_j}(b, a) \right]_{j,k} = \det DF(a) \neq 0.$$

By the implicit function theorem there exists a neighbourhood $W \ni b$ and $B = \mathbb{B}(a, \epsilon) \subset D$ such that for $(w, z) \in W \times B$ then

$$G(w, z) = 0 \iff z = H(w)$$

for some holomorphic $H : W \to B$. H is the inverse to F. □

3.3 Biholomorphic Equivalence of Domains

We would like to know when two domains have a biholomorphism from one to another. For $n = 1$ the Riemann Mapping Theorem states that if $\Omega \subset \mathbb{C}$ is a simply-connected domain then $\Omega \cong_{biholo} \mathbb{D}(0,1)$. This is also true for any simply connected open Riemann surface.

Exercise. Consider

$$\{ r_1 < |z| < R_1 \}, \{ r_2 < |z| < R_2 \}.$$

When are they biholomorphic?

For $n > 1$ there is no such nice result as the Riemann Mapping Theorem. In fact the two model domains $\mathbb{B}^n(0,1)$ and $\mathbb{D}^n(0,1)$ are not biholomorphic!

Theorem 3.3 (Poincare, 1907)**.** *If $n > 1$ then*

$$\mathbb{B}^n(0,1) \ncong_{biholo} \mathbb{D}^n(0,1).$$

Proof. Let $n = 2$, and write $(z,w) \in \mathbb{C}^2$. Assume that $\exists F = (f_1, f_2) : \mathbb{D}^2 \to \mathbb{B}^2$ biholomorphic.

Claim. $\forall w \in \mathbb{D}$, the map $F_w : \mathbb{D} \to \mathbb{C}$ given by

$$F_w(z) = \left(\frac{\partial f_1}{\partial w}(z,w), \frac{\partial f_2}{\partial w}(z,w) \right)$$

satisfies

$$\lim_{z \to b\mathbb{D}} F_w(z) = 0.$$

Note that this claim implies the theorem, as F_w extends continuously to $\overline{\mathbb{D}}$ and $\equiv 0$ on $b\mathbb{D}$. This cannot happen unless $F_w \equiv 0$, and so F is independent of w. This means that F cannot be biholomorphic.

Proof of Claim. It is enough to show that $\forall \{ z_\gamma \} \subset \mathbb{D}$ with $|z_\gamma| \to 1$, there exists a subsequence $\{ z_{\gamma_j} \}$ such that

$$\lim_{j \to \infty} F_w(z_{\gamma_j}) = 0.$$

Given such a sequence, apply Montel's theorem to the sequence

$$\{ F(z_\gamma, \cdot) \}, \qquad F(z_\gamma, \cdot) : \mathbb{D} \to \mathbb{B}^2$$

to get the holomorphic function $\phi : \mathbb{D} \to \overline{\mathbb{B}^2}$ as the limit. Since F is biholomorphic, $F(z_\gamma, w) \to b\mathbb{B}^2$ as $z_\gamma \to b\mathbb{D}$ for any $w \in \mathbb{D}$. Thus $\phi(\mathbb{D}) \subset b\mathbb{B}^2$. If $\phi = (\phi_1, \phi_2)$ then $\forall w \in \mathbb{D}$ we have

$$|\phi_1(w)|^2 + |\phi_2(w)|^2 = 1.$$

Applying the operator $\frac{\partial^2}{\partial w \partial \overline{w}}$ to both sides of this equation gives us

$$|\phi_1'(w)|^2 + |\phi_2'(w)|^2 = 0.$$

Thus $\phi'(w) = 0$. Since $F_w(z_\gamma) \to \phi'(w)$ then this proves the claim. □

Then we are done. □

Note. We can repeat this same argument for $n > 2$ without much complexity, just more space.

Remark. Unlike one variable, very few domains in several complex variables are biholomorphic, and it is quite notable if they are.

Definition. We write $D \cong D'$ and say that D and D' are **biholomorphically equivalent** if there exists a biholomorphic map $F : D \to D'$.

Non-Example 3.1. $\mathbb{B}^n(0,1) \ncong \mathbb{D}^n(0,1)$. ◁

Definition. Let $D \subset \mathbb{C}^n$ be a domain. Then a biholomorphism $f : D \to D$ is called a (biholomorphic) **automorphism of** D. Define the **automorphism group of** D as

$$\operatorname{Aut}(D) = \{ \text{biholomorphic automorphisms of } D \} .$$

This is a group under composition.

Remark. If $D \cong D'$ then $\operatorname{Aut}(D) \cong_{iso} \operatorname{Aut}(D')$.

Remark. It is true that if $G \in \operatorname{Aut}(\mathbb{B}^n)$ then $G = U \circ L_a$ for some $a \in \mathbb{B}^n$, where U is a unitary transformation and L_a is a biholomorphic linear fractional map $\mathbb{B}^n \to \mathbb{B}^n$ that sends a to 0.

add picture

More specifically let l_a be the complex line that passes through 0 and a. Denote $z_a' = \operatorname{proj}_{l_a} z$ and $z_a'' = z - z_a'$. Then L_a is the map

$$L_a : z \mapsto \frac{a - z_a' - \sqrt{1 - a^2} z_a''}{1 - \langle z, a \rangle}.$$

It is true that

$$\operatorname{Aut}(\mathbb{B}^n) \subset \{ \text{linear fractional transformations on } \mathbb{C}^n \} .$$

It is further true that $\operatorname{Aut}(\mathbb{B}^n)$ is a group depending on $n^2 + 2n$ real parameters.

Remark. We have that

$$\operatorname{Aut}(\mathbb{D}^n) = \{ z_j \mapsto e^{i\theta_{\sigma(j)}} \left[\frac{z_{\sigma(j)} - a_{\sigma(j)}}{1 - \overline{a_{\sigma(j)}} z_{\sigma(j)}} \right] \}$$

where σ runs over permutations of $\{ 1, \cdots, n \}$. This depends on $3n$ real parameters. It follows that

- if $n = 1$ then $\operatorname{Aut}(\mathbb{B}^n) \cong \operatorname{Aut}(\mathbb{D}^n)$,
- if $n > 1$ then $\operatorname{Aut}(\mathbb{B}^n) \ncong \operatorname{Aut}(\mathbb{D}^n)$.

Thus $\mathbb{B}^n \ncong \mathbb{D}^n$.

Remark. For $n = 1$ we have a property in the Riemann Mapping Theorem that lets us map a to b.

Add picture

For $n > 1$ we have a similar result. For $\mathbb{B}^n, \mathbb{D}^n$ then $\forall a \exists f \in \operatorname{Aut}$ such that $f(a) = 0$. This implies that $\operatorname{Aut}(\mathbb{B}^n), \operatorname{Aut}(\mathbb{D}^n)$ are transitive, ie $\forall a, b$ there exists $f \in \operatorname{Aut}$ such that $f(a) = b$.

Theorem 3.4 (Bedford Dadok, 1972). *For every real linear Lie group L there exists a domain $D \subset \mathbb{C}^n$ such that* $\operatorname{Aut}(D) \cong L$.

3.4 Automorphisms of $\mathbb{C}^n$

We would like to classify the automorphism group of $\mathbb{C}^n$.

$n = 1$: If $F : \mathbb{C} \to \mathbb{C}$ is biholomorphic then $F(z) = az + b$ (this is affine), where $a, b \in \mathbb{C}$. Thus

$$\dim_{\mathbb{R}} \operatorname{Aut}(\mathbb{C}) = 4.$$

$n > 1$: It is true that

$$\dim_{\mathbb{R}} \operatorname{Aut}(\mathbb{C}^n) = \infty.$$

Example 3.1. Define the map $f(z, w) = (z, w + \phi(z))$ where $\phi \in \mathcal{O}(\mathbb{C})$ is entire. Then $f \in \operatorname{Aut}(\mathbb{C}^2)$ as $f^{-1}(z', w') = (z', w' - \phi(z'))$. These are called "shears" and are dense in $\operatorname{Aut}(\mathbb{C}^n)$. ◁

Remark. Biholomorphicity does not in general imply surjectivity.

$n = 1$: Consider $F : \mathbb{C} \to \mathbb{C}$ biholomorphic on its image. Then F is surjective.

$n > 1$: There exists a map $F : \mathbb{C}^n \to \mathbb{C}^n$ which is injective and biholomorphic onto its image. However $F(\mathbb{C}^n) \neq \mathbb{C}^n$. In fact $\mathbb{C}^n \setminus F(\mathbb{C}^n)$ has non-empty interior. The images of such F are called **Fatou-Bieberbach domains**.

Proposition 3.1 (Stensones, 2000)**.** *There exists a map $F : \mathbb{C}^2 \to \mathbb{C}^2$ such that $F(\mathbb{C}^2)$ has smooth boundary.*

Note. For $\mathbb{C}^2$ define

$$\zeta(z_1, z_2) = (z_2, a^2 z_1 - (1 - a) z_2^2).$$

Then the basin of attraction of 0 is the set

$$\{ z \in \mathbb{C}^2 \mid \lim_{\gamma \to \infty} \zeta^{\circ \gamma}(z) = 0 \}.$$

3.5 The Jacobian Conjecture

Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a complex linear function. This can be seen as a "polynomial of degree one". Then

$$\det DA \neq 0 \iff A \in \operatorname{Aut}(\mathbb{C}^n).$$

We want to try to extend this to polynomials of higher degree. Let $F = (f_1, \cdots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial, that is let f_j be a polynomial in $z_1, \cdots, z_n$. Then $\det DF$ is a polynomial. Then there are two possible cases:

- $\det DF$ is non-constant or identically 0. Then by the fundamental theorem of algebra there exists some $z \in \mathbb{C}^n$ such that $\det DF(z) = 0$. Then F is not locally invertible and is not injective.
- $\det DF$ is a non-zero constant.

Conjecture (Keller, 1939)**.** If $\det DF = cst \neq 0$ then F is invertible and F^{-1} is a polynomial. Thus $F \in \operatorname{Aut}(\mathbb{C}^n)$.

Remark. The Jacobian conjecture is false for entire maps. For example consider $F : (z, w) \mapsto (e^z, e^{-z}w)$. Then

$$\det DF = \det \begin{bmatrix} e^z & 0 \\ -e^{-z}w & e^{-z} \end{bmatrix} = e^z e^{-z} = 1,$$

but F is not injective, since

$$F(0,0) = (1,0) = F(2\pi i, 0).$$

Remark. The Jacobian conjecture fails as well for Fatou Bieberbach type functions, ie maps which are injective and biholomorphic onto their image, but fail to be surjective.

Remark. The Jacobian Conjecture holds if $\deg F = 2$ in any dimension.

Remark. If the Jacobian Conjecture holds for polynomials of degree 3 in all dimensions then it holds in general.

Theorem 3.5. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial, and suppose DF is a non-zero constant. Then the following are equivalent:*

1. *F is invertible and F^{-1} is a polynomial.*
2. *F is injective.*
3. *F is proper, ie $\forall K \Subset \mathbb{C}^n$ compact then $F^{-1}(K)$ is a compact subset of $\mathbb{C}^n$.*

Remark. The Jacobian Conjecture cn an be formulated for any field $\mathbb{K}$, not just $\mathbb{C}$.

Remark. The Jacobian Conjecture is false for fields with finite characteristic.

Remark. If the Jacobian Conjecture holds for $\mathbb{C}$ then it holds for any algebraically closed field of characteristic 0.

Remark. What about $\mathbb{K} = \mathbb{R}$? Then either $DF \neq 0$ or DF is a non-zero constant.

Non-Example 3.2. There exists a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ with $DF = cst \neq 0$ such that F is not invertible. Note that $\deg F \approx 100$. ◁

3.6 Complex Manifolds

Definition (Real Manifolds)**.** Let M be a Hausdorff topological space. M is a **real manifold** if M is covered by coordinate charts (U, ϕ) where $U \subset M$ and $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a homeomorphism, such that

$$\phi_V \circ \phi_U^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$

is smooth. $\{ (U, \phi) \}$ is called an **atlas**. add picture

Definition. Complex manifolds are the same, except replace $\mathbb{R}^n$ with $\mathbb{C}^n$ and instead of smooth transition maps require biholomorphic transition maps.

Example 3.2. Open subsets of $\mathbb{C}^n$ are complex manifolds. ◁

Example 3.3. The complex projective space $\mathbb{CP}^n$ is a complex manifold. To construct this consider $\mathbb{C}^n$ and write $z = (z_1, \cdots, z_n)$. Introduce homogeneous coordinates

$$z_j = \frac{w_j}{w_0}, \quad w_0 \neq 0.$$

This gives a correspondence $z \to w = (w_0, w_1, \cdots, w_n)$ where $w_0 \neq 0$. This is $1-1$ up to scalar multiplication by $\lambda \in \mathbb{C} \setminus \{0\}$. Then define the quotient

$$\mathbb{CP}^n = \{\, w = (w_0, w_1, \cdots, w_n) \neq 0 \,\} \Big/ w' = \lambda w'' \text{ for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

Then we have that

$$\mathbb{C}^n_z \cong \mathbb{CP}^n \cap \{\, w_0 \neq 0 \,\}.$$

Note as well that the points at infinity for $\mathbb{C}^n$ is

$$\mathbb{CP}^n \cap \{\, w_0 = 0 \,\} \cong \mathbb{CP}^{n-1}.$$

Then $\mathbb{CP}^n$ is a compact complex manifold with atlas $(U_j, \phi_j), j = 0, \cdots, n$ where

$$U_j = \mathbb{CP}^n \cap \{\, w_j \neq 0 \,\}.$$

We write $[w]$ for the equivalence class of w. Then

$$\phi_j[w] = \left(\frac{w_0}{w_j}, \cdots, \frac{w_{j-1}}{w_j}, \frac{w_{j+1}}{w_j}, \cdots, \frac{w_n}{w_j} \right),$$
$$\phi_j : U_j \to \mathbb{C}^n.$$

◁

Exercise. Show that $\phi_j \circ \phi_k^{-1}$ is biholomorphic onto its image where defined.

Example 3.4. Note that

$$\mathbb{CP}^n = \{\, \text{complex lines passing through 0 in } \mathbb{C}^{n+1} \,\}.$$

Replacing complex lines with k-dimensional complex linear subspaces to get the complex Grassmannian

$$G(n,k) = \{\, k\text{-dimensional complex subspaces of } \mathbb{C}^n \,\},$$

where $0 < k < n$. This is a compact complex manifold of dimension $k(n-k)$. Note that

$$G(1,n) \cong \mathbb{CP}^{n-1}.$$

◁

Example 3.5. Complex submanifolds of $\mathbb{C}^n$ are complex manifolds. ◁

Definition. $X \subset \mathbb{C}^n$ is a **complex submanifold** of $\mathbb{C}^n$ if $\forall p \in X$ there exists a neighbourhood $p \in U \subset \mathbb{C}^n$ and a biholomorphism $\phi : U \to \mathbb{C}^n_w$ such that

$$\phi(X \cap U) = \phi(U) \cap \{\, w_1 = \cdots = w_k = 0 \,\}.$$

We say that $\dim_{\mathbb{C}} X = n - k$. Note that (U, ϕ) are coordinate charts on X.

Exercise. $X \subset \mathbb{C}^n$ is a complex submanifold of dimension d if $\forall p \in X$ there exists a neighbourhood U of p such that $U = U' \times U''$ and $U \cap X$ is the graph of some holomorphic map $f : U' \to U''$.

add picture.

Theorem 3.6. *Let $D \subset \mathbb{C}^n$, $F : D \to \mathbb{C}^m$ be a non-singular (ie $DF(z)$ has full rank for all z) holomorphic map. Then $\forall a \in \mathbb{C}^m$, $F^{-1}(a)$ is a complex submanifold of* $\dim = \max\{ n - m \}$. *If $m \geq n$ then $F^{-1}(a)$ is either a point (a 0-dimensional manifold) or $F^{-1}(a) = \emptyset$.*

Proof. $F^{-1}(a)$ is a smooth manifold. Without loss of generality then $n > m$. Replace F with $\tilde{F} = F - F(a)$. After relabeling coordinates $\tilde{F}$ satisfies the assumptions of the implicit function theorem. Thus there exists a neighbourhood U or $p \in F^{-1}(a)$ such that $\{ z \in U \mid \tilde{F} = 0 \} =$ the graph of a holomorphic function $h : U' \to U''$ where $U' \subset \mathbb{C}^{n-m}$ and $U'' \subset \mathbb{C}^m$. Thus h is a local parameterization of $F^{-1}(a)$. □

Remark. Let $X \subset \mathbb{C}^n$ be a complex submanifold of dimension $0 < k < n$. Then for $p \in X$, $T_pM \cong \mathbb{C}^k$.

add picture

The following theorem is converse to the previous remark.

Theorem 3.7 (Levi Civita). *Suppose $X \subset \mathbb{C}^n$ is a real submanifold of even dimension $2k$. Suppose $\forall p \in X$ then $T_pX \cong \mathbb{C}^k$. Then X is a complex submanifold of $\mathbb{C}^n$.*

Example 3.6. Let $f : \mathbb{C}^n \to \mathbb{C}$ be C^1 smooth. Then the graph of f is a smooth manifold.

add picture

If $T_p\Gamma_f \cong \mathbb{C}^n \subset \mathbb{C}^{n+1}$ then by Levi Civita Γ_f is a complex submanifold of $\mathbb{C}^{n+1}$. This implies that f is holomorphic. Note that the condition on $T_p\Gamma_f$ is equivalent to saying that f is $\mathbb{C}$-linear at each point. ◁

3.7 Holomorphic Functions and Maps on Complex Manifolds

We want to define holomorphicity of functions from manifolds to $\mathbb{C}^n$.

Definition. Let X be a complex manifold of (complex) dimension k. A function $f : M \to \mathbb{C}^m$ is **holomorphic** if $f \circ \phi^{-1}$ is holomorphic for any coordinate chart (U, ϕ) on X.

Now suppose that $X \subset \mathbb{C}^n$ is a submanifold and $g : U' \to U''$ where $U' \subset \mathbb{C}^k$ and $U'' \subset \mathbb{C}^{n-k}$ is a local parameterization.

Definition. We say that f is **holomorphic** if $f(z', g(z'))$ is a holomorphic map on U' for any parameterization g.

Remark. Various complex analytic results, such as the maximum principle and the uniqueness theorem, have manifold analogues.

We have two propositions.

Proposition 3.2. *If X us a compact connected complex manifold then any function that is holomorphic on X is a constant.*

Proof. $f \in \mathcal{O}(X) \implies f \in C(X)$, so f attains global max at $p \in X$. Apply the max principle to f on some small neighbourhood p in X to achieve our result. □

Proposition 3.3. *If M is a compact complex submanifold on $\mathbb{C}^n$, then M is a finite union of points.*

Proof. Let $\tilde{M} \subset M$ be a connected component of M. Consider the function $\pi_j|_{\tilde{M}}$, ie projection onto the j-th coordinate. This is a holomorphic function on $\tilde{M}$. By the previous proposition $\pi_j|_{\tilde{M}}$ is constant. Since j is arbitrary then $\tilde{M}$ is a point. □

3.8 Complex Analytic Sets

Definition. A set $A \subset \mathbb{C}^n$ is a **local complex analytic set** if $\forall p \in A$ there exists and neighbourhood $p \in U \subset \mathbb{C}^n$ and $f_1, \cdots, f_k \in \mathcal{O}(U)$ such that

$$A \cap U = \{ f_1 = \cdots = f_k = 0 \} .$$

Definition. A set $A \subset \Omega \subset \mathbb{C}^n$ is a **complex analytic set** if $\forall p \in \Omega$ there exists a neighbourhood $p \in U \subset \mathbb{C}^n$ and $f_1, \cdots f_k \in \mathcal{O}(U)$ such that

$$A \cap U = \{ f_1 = \cdots = f_k = 0 \} .$$

Remark. We write "caset" to mean complex analytic set. In some literature local casets are called complex analytic sets, and casets are called complex analytic subsets.

Example 3.7.

$n = 1$:
- The local casets are either points or domains.
- The casets are just discrete sets of points.

This is the case for local casets since if we select a p inside a domain we can use the 0 function. This is the case for casets since otherwise we can pick something on the boundary of A to get a contradiction.

$n > 1$:
- Any open set is a local caset but not a caset.
- $\{ 0 \} = \{ z_j = 0 \mid j = 1, \cdots, n \}$ is a caset.
- $\{ z_n = 0 \}$ is a caset. In fact any complex submanifold is a caset.
- $\{ z_1^2 = z_2^3 \} \subset \mathbb{C}^2$ is a caset but not a complex submanifold of $\mathbb{C}^2$ (cusp at 0).

◁

Definition. A set A is **reducible** if $A = A_1 \cup A_2$ where A_1, A_2 are different casets. A is **irreducible** if $A = A_1 \cup A_2$ implies $A_1 = A$ or $A_2 = A$.

Proposition 3.4. *Finite unions and finite intersections of casets are casets.*

Proof. Let $A = \{ f_1 = \cdots f_k = 0 \}$ and $B = \{ g_1 = \cdots = g_l = 0 \}$ locally near $p \in \mathbb{C}^n$. Then

$$A \cup B = \{ f_\mu g_\nu = 0 \mid \mu \in [k], \nu \in [l] \} ,$$
$$A \cap B = \{ f_1 = \cdots f_k = g_1 = \cdots = g_l = 0 \}$$

locally near p. □

Proposition 3.5. *Let $A \subset \Omega$ be a caset with $A \neq \Omega$. Then A is nowhere dense in Ω.*

Proof. If A is dense in some $U \subset \Omega$ then locally

$$A = \{ f_1 = \cdots f_k = 0 \} ,$$

where f_j vanish on a dense set in U. This implies that

$$f_j|_U \equiv 0$$

and so since Ω is connected then this forces $A = \Omega$. □

Definition. A point $z \in A$ is a **regular point** of a caset A if near z then A is a complex manifold. Otherwise z is a **singular point**. We write

$$\begin{aligned} A^{reg} &= \{ z \in A \mid z \text{ is a regular point of } A \}, \\ A^{sng} &= A \setminus A^{reg}. \end{aligned}$$

Example 3.8. Let $A = \{ z_1^2 - z_2^3 = 0 \} \subset \mathbb{C}^2$. Then $A = \{ f^{-1}(0) \mid f(z_1, z_2) = z_1^2 - z_2^3 \}$. Then $Df(z) = (2z_1, -3z_2)$. Then $Df(z) = 0$ iff $z = 0$. Thus $A \setminus \{0\}$ is a complex manifold, and so $A^{reg} = A \setminus \{0\}$ and $A^{sng} = \{0\}$. Note that A is an irreducible caset. ◁

Example 3.9. Let $A_1 = \{ z_3 = 0 \}$ and $A_2 = \{ z_1 = z_2 = 0 \}$.

add picture

Then $A = A_1 \cup A_2$ is a caset. If $z \neq 0$ then near z A is a complex submanifold of $\dim_{\mathbb{C}} = 1$ or $\dim_{\mathbb{C}} = 2$. It is true that $A^{reg} = A \setminus \{0\}$ and $A^{sng} = \{0\}$. Note as well that A is a reducible caset. ◁

Definition. Let A be a caset. If $z \in A^{reg}$ then we define $\dim_z A = \dim_{\mathbb{C}} A$ as a complex submanifold near z. If $z \in A^{sng}$ then we define

$$\dim_z A = \limsup_{\substack{w \to z \\ w \in A^{reg}}} \dim_w A.$$

Example 3.10. In the first example then $\dim_z A = 1$ fr every z. In the second example then $\dim_z A = 2$ if $z_3 = 0$ and $\dim_z A = 1$ otherwise. ◁

Definition. Say $\dim A = \max_{z \in A} \dim_z A$.

Proposition 3.6. *Let $D \subset \mathbb{C}^n$, $f \in \mathcal{O}(D)$ with $f \not\equiv 0$, and let $A = \{ f = 0 \} \neq \emptyset$. Then A is a caset in D and* $\dim A = n - 1$.

Proof. A is nowhere dense in D, which implies that $\dim A \leq n - 1$. Take any $z \in A$. By the uniqueness theorem $\exists j = (j_1, \cdots, j_n)$ with $|j| \geq 0$ such that

$$g := \left. \frac{\partial^{|j|} f}{\partial z^j} \right|_A \equiv 0$$

but $dg|_A \not\equiv 0$. Thus there exists $b \in A^{reg}$ near z so $A \cap U_b$ (where U_b is a neighbourhood of b) is a complex submanifold of dimension $n - 1$. Since z is arbitrary and b is arbitrarily close to z then $\dim A = n - 1$. □

Exercise. There is an issue with this proof! Find it.

Note. The issue with this proof is that g might have a larger vanishing set than just A (ie $A \subsetneq \{ g = 0 \}$), and so we can only say that $\dim A \leq n - 1$, not that $\dim A = n - 1$.

Remark. Note that A^{reg} is open in A.

Theorem 3.8. *$A^{reg} \subset A$ is dense.*

Proof. We induct on the dimension n. If $n = 1$ then A is a union of points and is thus a complex manifold of dimension 0. Then $A = A^{reg}$, which suffices.

Now suppose that this is true for $n - 1$. Let $a \in A$, and let $U \ni a$ be a neighbourhood such that

$$U \cap A = \{ f_1 = \cdots f_N = 0, \; f_j \in \mathcal{O}(U) \}.$$

It suffices to prove that $U \cap A^{reg} \neq \emptyset$. Wlog $f_1 \not\equiv 0$. Then $f_1|_{A\cap U} = 0$. Then there exists an index j_0 such that

$$g = \left.\frac{\partial^{|j_0|} f_1}{\partial z^{j_0}}\right|_{A\cap U} \equiv 0$$

but some derivative

$$\frac{\partial g}{\partial z_k}(b) \neq 0$$

for some $b \in A \cap U$. By the implicit function theorem $\exists V \ni b$ with $V \subset U$ such that

$$A_g = \{\, z \in V \mid g(z) = 0 \,\}$$

is a complex submanifold of V of dimension $n-1$. $g|_{V\cap A} = 0$, and so $A \cap V \subset A_g$. Since A_g is a complex submanifold of dimension $n-1$ then $A \cap V$ contains a regular point by the induction hypothesis, and so we are done. □

This shows that our notion of dimension for casets is well-defined.

Remark. It is true that A^{sng} is a caset itself, with $\dim A^{sng} < \dim A$. This means we can somehow decompose a caset into a union of manifolds of different dimensions.

This is all the preparation we need to for the following important theorem due to Weierstrass (though Siegel disagreed). This theorem is a generalization of the implicit function theorem and is a very useful tool when dealing with several complex variables.

Theorem 3.9 (Weierstrass Preparation Theorem)**.** *Let $f \in \mathcal{O}(V)$ where*

$$V = V' \times \{|z_n| < R\,\} \subset \mathbb{C}^{n-1} \times \mathbb{C} = \mathbb{C}^n,$$

where V' is a neighbourhood of $0'$ in $\mathbb{C}^{n-1}$. Suppose that $f(0', z_n) \not\equiv 0$ in $|z_n| < R$. Let $0 < r < R$ be such that $f(0', z_n) \neq 0$ on $|z_n| = r$. Let k be the number of 0's of $f(0', z_n)$ in $\{|z_n| < r\,\}$ (counting multiplicities).

Then in some neighbourhood $U = U' \times U_n \subset V$ of $0 \in \mathbb{C}^n$ we have

$$f(z) = \left(z_n^k + c_1(z')z_n^{k-1} + \cdots c_k(z')\right)\phi(z) = \psi(z)\phi(z)$$

where $\phi(z) \in \mathcal{O}(U)$, $\phi(z) \neq 0$, and $c_j(z') \in \mathcal{O}(U')$.

Note. In the previous theorem we call $\psi(z)$ the distinguished polynomial (or the Weierstrass pseudopolynomial) of f.

Remark. Note that if $n = 1$ then $f(z_0) = 0$ implies that we can write $f(z) = (z - z_0)^k \phi(z)$ where $\phi(z_0) \neq 0$. Note also that f might vanish on the z_n axis, but we can assume not after some change of variables.

Proof. Write $U_n = \{|z_n| < r\,\}$. Since $f(0', z_n) \neq 0$ on $\{|z_n| = r\,\}$, there exists $U' \ni 0'$ such that $\overline{U'} \times \overline{U_n} \subset V$ such that $f(z', z_n) \neq 0$ on $U' \times \{|z_n| = r\,\}$. Let $n(z')$ be the number of 0's of $f(z', z_n)$ for $z' \in U'$ fixed in U_n. By the residue theorem in one variable we can write

$$n(z') = \frac{1}{2\pi i}\int_{|z_n|=r} \frac{\partial f(z', z_n)}{\partial z_n} \cdot \frac{1}{f(z', z_n)}\, \mathrm{d}z_n.$$

Then $n(z')$ is continuous on U' and takes only integer values. Thus $n(z') = k$ is constant on U'.

add image

Note that zeroes form paths that can intersect. This is okay since we count multiplicites of 0's.

Now let $\alpha_1(z'), \cdots, \alpha_k(z')$ be the 0's of $f(z', z_n)$ counting multiplicities. Write this in any order. Set

$$P(z', z_n) = \prod_{j=1}^{k}(z_n - \alpha_j(z')) = z_n^k + c_1(z')z_n^{k-1} + \cdots + c_k(z').$$

Claim. $c_j(z')$ are holomorphic on U'.

Proof of Claim. By Vieta's theorem, if we write

$$p(z) = (z - z_0)\cdots(z - z_k) = z^k + c_1 z^{k-1} + \cdots + c_k$$

then c_j is a symmetric function of the roots. For example in $(x-a)(x-b) = x^2 - (a+b)x + ab$, $a+b$ and ab are symmetric in a, b. A general fact from algebra says that elementary symmetric functions have an "elementary symmetric basis" given (if $x = (x_1, \cdots, x_n)$ by

$$\sigma_m(x) = \sum_{j=1}^{n} x_j^m.$$

Combining these two give that $c_J(z') = g_j\left[\sigma_1(\alpha(z')), \cdots \sigma_k(\alpha(z'))\right]$ where g_j is a polynomial and $\sigma_j(\alpha(z'))$ is an elementary symmetric polynomial in $\alpha_1, \cdots, \alpha_k$. To prove that $c_j(z') \in \mathcal{O}(U')$ it suffices to prove that $\sigma_j(\alpha(z'))$ is holomorphic, ie prove that $S_m(z') = \sum_{j=1}^{k} \alpha_j(z')^m$ are holomorphic.

Recall from single-variable complex analysis that if g and h are holomorphic in $|\zeta| \leq r$ and $g \neq 0$ on $|\zeta| = r$ then

$$\frac{1}{2\pi i}\int_{|\zeta|\leq r} h(\zeta)\frac{g'(\zeta)}{g(\zeta)}\,\mathrm{d}\zeta = \sum_{j=1}^{N} h(\alpha_j)$$

where $\{\alpha_j\} = g^{-1}(0)$. Replacing ζ by z_n, $h(\zeta)$ by z_n^m, and $g(\zeta)$ by $f(z', z_n)$ for some fixed $z' \in U'$ gives us that

$$\begin{aligned} S_m(z') &= \sum_{j=1}^{k} \alpha_j(z')^m \\ &= \frac{1}{2\pi i}\int_{\mathrm{b}U_n} z_n^m \left(\frac{\partial f(z', z_n)}{\partial z_n} \cdot \frac{1}{f(z', z_n)}\right) \mathrm{d}z_n. \end{aligned}$$

It is a fact that $F(z,t)$ is C^1-smooth in a neighbourhood of $\Omega \times K \subset \mathbb{C}^n \times \mathbb{R}^m$ with K compact, and if $F(z,t)$ is holomorphic in z for any $t \in K$ fixed, then letting μ be any measure on $\mathbb{R}^m$ we have

$$f(z) = \int_K F(z,t)\,\mathrm{d}\mu(t) \in \mathcal{O}(\Omega).$$

For the proof of this compute $\frac{\partial f(z)}{\partial \bar{z}_j}$ using the definition of $\frac{\partial}{\partial \bar{z}_j}$. From this fact, we have that $S_m(z')$ is holomorphic, and we are done. □

We are now done. □

Remark.

- If $f(0) = 0$ in the WPT then $c_j(0) = 0$ for all j.
- Let $A = \{ f = 0 \}$ with f as in the WPT. Let $\pi : A \to U'$ be coordinate projection. Then π is a branched analytic covering, ie there exists a caset $E \subset U'$ (discriminant set) such that

$$\pi\Big|_{A\setminus\pi^{-1}(E)} : A \setminus \pi^{-1}(E) \to U' \setminus E.$$

Note as well that $A \setminus \pi^{-1}(E) \subset A^{reg}$. This is called a k-sheeted covering of $U' \setminus E$.

Theorem 3.10 (Weierstrass Division Theorem). *Let $f \in \mathcal{O}(U)$ with $0 \in U$. Let P be a Weierstrass pseudopolynomial in z_n in a neighbourhood of 0 (note that P is not necessarily the Weierstrass pseudopolynomial of f). Then in a possibly smaller neighbourhood of 0 we have that*

$$f = P \cdot \phi + Q$$

where ϕ is holomorphic near 0 and Q is a pseudopolynomial with $\deg Q < \deg P$.

3.9 Review of Differential Forms

Definition. on $\mathbb{R}^n$ then

$$\omega = \sum_{|j|=d} f_j \, \mathrm{d}x^j$$

where $\mathrm{d}x^j = \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_n}$ is a **differential form** of degree d.

Remark.

- If ω is a degree 0 form then ω is a function.
- If ω is a degree n form then $\omega = f \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_1$.

This last point is because the $\wedge$ operation is anticommutative, ie $\mathrm{d}x_a \wedge \mathrm{d}x_b = -\,\mathrm{d}x_b \wedge \mathrm{d}x_a$.

Remark. Differential forms exist to be integrated. Consider an n-form ω on $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \int_{\mathbb{R}^n} f \, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

where this is now “normal" integration.

Definition. Define Λ^k to be all differential forms of degree k for some $0 \leq k \leq n$. Then we define the **exterior derivative** as the map

$$\mathrm{d} \colon \Lambda^k \to \Lambda^{k+1}$$

with

$$\mathrm{d}\left(\sum_{|j|=k} f_j \, \mathrm{d}x_j \right) = \sum_{|j|=k} \mathrm{d}f_j \wedge \mathrm{d}x_j$$

where

$$\mathrm{d}f = \sum_{\nu=1}^{n} \frac{\partial f}{\partial x_\nu} \, \mathrm{d}x_\nu.$$

Note that here f is a 0-form and $\mathrm{d}f$ is a 1-form. In particular we have the 1-forms $\mathrm{d}x_\nu$ for $\nu = 1, \cdots, n$. Think of these as differentials of coordinate functions.

Remark. It is true that

- $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$.
- $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{\deg(\omega_1)}\omega_1 \wedge \mathrm{d}\omega_2$.
- $\mathrm{d}^2 = 0$

Consider now a map $(\mathbb{R}^n, x) \xrightarrow{\phi} (\mathbb{R}^m, x')$ where ω is an n-form on $\mathbb{R}^m$ and $\phi(\mathbb{R}^n)$ is an n-dimensional manifold in $\mathbb{R}^m$. Then we can integrate ω on $\phi(\mathbb{R}^n)$. To do this we defined the pullback.

Definition. Let

$$\omega = \sum_{|j|=n} f_j \,\mathrm{d}(x')^j.$$

The **pullback** of ω to $\mathbb{R}^n$ is

$$\phi^*\omega = \sum_{|j|=n} \left(f_j \circ \phi\right) \mathrm{d}\big(\phi(x)\big)^j.$$

Example 3.11. If $\omega = \mathrm{d}x_1' \wedge \mathrm{d}x_2'$, $\phi(x) = \big(\phi_1(x), \cdots, \phi_n(x)\big)$ then $\phi^*\omega = \mathrm{d}\big(\phi_1(x)\big) \wedge \mathrm{d}\big(\phi_2(x)\big)$. ◁

Then $\phi^*\omega$ is an n-form on $\mathbb{R}^n$. Then $\int_{\mathbb{R}^n} \phi^*\omega$ is well-defined.

$$\int_{\phi(\mathbb{R}^n)} \omega := \int_{\mathbb{R}^n} \phi^*\omega.$$

Definition. A form ω is

- **closed** if $\mathrm{d}\omega = 0$,
- **exact** if $\exists \alpha$ such that $\omega = \mathrm{d}\alpha$.

Note that any exact form is also closed.

Remark. We can do deRham cohomology from here.

Theorem 3.11 (Stokes). *let $\Omega \subset \mathbb{R}^n$ be a domain with piecewise smooth boundary, and let ω is an $(n-1)$-form. Then*

$$\int_{\mathrm{b}\Omega} \omega = \int_\Omega \mathrm{d}\Omega.$$

Note that we can pull back $\int_{\mathrm{b}\Omega} \Omega$ to $\mathbb{R}^{n-1}$.

Example 3.12. On $[a, b] \subset \mathbb{R}$ we have

$$\underbrace{f(b) - f(a)}_{\text{integral over}\{a,b\}} = \int_{(a,b)} \mathrm{d}f = \int_a^b f'(x)\,\mathrm{d}x.$$

◁

3.10 Differential Forms on $\mathbb{C}^n$.

Most things apply since $\mathbb{C}^n \approx \mathbb{R}^{2n}$. Write

$$z_j = x_j + iy_j.$$

The building blocks are now $\mathrm{d}x_j$ and $\mathrm{d}y_j$. Note we can write any differential form in $\mathrm{d}x_j$ and $\mathrm{d}y_j$ in terms of $\mathrm{d}z_j$ and $\mathrm{d}\bar{z}_j$ instead. It follows that any k-form ω on $\mathbb{C}^n$ can be written as

$$\omega = \sum_{|\nu|+|\mu|=k} f_{\nu\mu}\, \mathrm{d}z^{\nu} \wedge \mathrm{d}\bar{z}^{\mu}$$

with

$$\begin{aligned} \nu &= (\nu_1, \cdots, \nu_n), \\ \mu &= (\mu_1, \cdots, \mu_n), \\ \mathrm{d}z^{\nu} &= \mathrm{d}z_{\nu_1} \wedge \cdots \wedge \mathrm{d}z_{\nu_n}, \\ \mathrm{d}\bar{z}^{\nu} &= \mathrm{d}\bar{z}_{\nu_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{\nu_n}. \end{aligned}$$

Thus any differential form ω can be written as a sum of bidegree (k, l).

Example 3.13. The differential form $\omega = \mathrm{d}z_1 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_5 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}\bar{z}_2$ has bidegree $(3, 2)$. ◁

Definition. Let $\Lambda^{(k,l)}$ be the set of differential forms of bidegree (k, l). Then define the operator

$$\partial : \Lambda^{(k,l)} \to \Lambda^{(k+1,l)}$$

with

$$\partial\left(\sum_{\substack{|\nu|=k \\ |\mu|=l}} f_{\nu\mu}\, \mathrm{d}z_{\nu}\, \mathrm{d}\bar{z}_{\mu} \right) = \sum_{\substack{|\nu|=k \\ |\mu|=l}} \sum_{j=1}^{n} \frac{\partial f_{\nu\mu}}{\partial z_j}\, \mathrm{d}z_j \wedge \mathrm{d}z_{\nu} \wedge \mathrm{d}\bar{z}_{\nu}.$$

This is pronounced as the "del" operator. We have a similar operator

$$\bar{\partial} : \Lambda^{(k,l)} \to \Lambda^{(k,l+1)}$$

with

$$\bar{\partial}\left(\sum_{\substack{|\nu|=k \\ |\mu|=l}} f_{\nu\mu}\, \mathrm{d}z_{\nu}\, \mathrm{d}\bar{z}_{\mu} \right) = \sum_{\substack{|\nu|=k \\ |\mu|=l}} \sum_{j=1}^{n} \frac{\partial f_{\nu\mu}}{\partial \bar{z}_j}\, \mathrm{d}\bar{z}_j \wedge \mathrm{d}z_{\nu} \wedge \mathrm{d}\bar{z}_{\nu}.$$

This is pronouced as the "del-bar" operator.

Example 3.14. $\bar{\partial}\,[\bar{z}_1 z_2\, \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2] = z_2\, \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2$. ◁

Exercise. Show that

1. $\mathrm{d} = \partial + \bar{\partial}$.
2. $\partial^2 = \bar{\partial}^2 = 0$ (but $\partial\bar{\partial} \neq 0$).

Remark. A C^1-smooth function f on $\mathbb{C}^n$ is holomorphic if and only if f is a $\bar{\partial}$-closed 0-form.

3.11 Bochner-Martinelli Integral Formula

Recall the Cauchy integral formula:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{b_0\mathbb{D}^n(0,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

This is nice, but $b_0\mathbb{D}^n(0,r)$ is not the full boundary of our domain $\mathbb{D}^n(0,r)$. We would like an integral formula with two properties: it should go over the full boundary of a domain, and it should give us a function on the interior of the domain just from the boundary values.

Consider a differential form ω on $\mathbb{C}^n$ of bidegree $(n, n-1)$

$$\omega(z) = \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}\bar{z}_\nu}{|z|^{2n}}\, d\bar{z}[\nu] \wedge dz$$

where

$$\begin{aligned} |z|^{2n} &= (|z|^2)^n = \left(|z_1|^2 + \cdots + |z_n|^2\right)^n, \\ d\bar{z}[\nu] &= d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{\nu-1} \wedge d\bar{z}_{\nu+1} \wedge \cdots \wedge \bar{z}_n, \\ dz &= dz_1 \wedge \cdots \wedge dz_n. \end{aligned}$$

We write the following lemmas.

Lemma 3.2. $d\big(\bar{z}_\nu\, d\bar{z}[\nu]\big) = (-1)^{\nu-1}\, d\bar{z}$.

Proof. $d\big(\bar{z}_\nu\, d\bar{z}[\nu]\big) = d\bar{z}_\nu \wedge d\bar{z}[\nu] = (-1)^{\nu-1}\, d\bar{z}$. □

Lemma 3.3. *ω is a closed form on $\mathbb{C}^n \setminus \{0\}$.*

Proof. We write

$$d\omega = \sum_{\nu=1}^{n} (-1)^{\nu-1} \frac{\partial}{\partial \bar{z}_\nu}\left(\frac{\bar{z}_\nu}{|z|^{2n}}\right) d\bar{z}_\nu \wedge d\bar{z}[\nu] \wedge dz.$$

Now

$$\begin{aligned} \frac{\partial}{\partial \bar{z}_\nu}\left(\frac{\bar{z}_\nu}{|z|^{2n}}\right) &= \frac{|z|^{2n} - \bar{z}_\nu \frac{\partial}{\partial \bar{z}_\nu}\left[z_1\bar{z}_1 + \cdots + z_n\bar{z}_n\right]^n}{(|z|^{2n})^2} \\ &= \frac{|z|^{2n} - n|z|^{2n-2} z_\nu \bar{z}_\nu}{|z|^{4n}}. \end{aligned}$$

Thus

$$\begin{aligned} d\omega &= \sum_{\nu=1}^{n} \left(\frac{1}{|z|^{2n}} - \frac{n z_\nu \bar{z}_\nu}{|z|^{2n+2}}\right) d\bar{z} \wedge dz \\ &= \left[\frac{n}{|z|^{2n}} - n\frac{(|z_1|^2 + \cdots + |z_n|^2)}{|z|^{2n+2}}\right] d\bar{z} \wedge dz \\ &= \left[\frac{n}{|z|^{2n}} - n\frac{|z|^2}{|z|^{2n+2}}\right] d\bar{z} \wedge dz \\ &= 0. \end{aligned}$$

Noting that ω isn't defined at 0 gives the result. □

Lemma 3.4. *$f\omega$ is a closed $(n, n-1)$-form.*

Proof. Take a sphere $S_r = \{|z| = r\}$. Note that ω is a $(2n-1)$-form as a real form. Then

$$\begin{aligned}\int_{S_r} \omega &= \frac{1}{r^{2n}} \int_{S_r} \sum_{\nu=1}^{n} (-1)^{\nu-1} \overline{z}_\nu \, d\overline{z}[\nu] \, dz \\ &= \frac{n}{r^2} \int_{B_r = \{|z|<r\}} d\overline{z} \wedge dz.\end{aligned}$$

The last equality follows by Stokes' theorem and Lemma 3.2.

In $\mathbb{R}$ we have a very solid notion of volume and dimensions. In $\mathbb{C}$ this is a bit more tricky. Write $dz_\nu = dx_\nu + i\, dy_\nu$. Then

$$d\overline{z}_\nu \wedge dz_\nu = 2i \, dx_\nu \wedge dy_\nu$$

and so

$$d\overline{z} \wedge dz = (2i)^n \, dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n.$$

Accepting a positive orientation, ie having $\int 1 \, dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \geq 0$ then

$$\int_{S_r} \omega = \frac{n}{r^{2n}} (2i)^n \int_{B_r} dx \wedge dy = \frac{(2\pi i)^n}{(n-1)!}.$$

noting that $\int_{B_r} dx \wedge dy = \frac{\pi^n r^{2n}}{n!}$ is the volume of an n-dimensional ball.

Now let $D \subset \mathbb{C}^n$ be a domain with piecewise smooth boundary that contains 0. Say $f \in \mathcal{O}(D) \cap C(\overline{D})$. Then for $z \neq 0$

$$d(f\omega) = df \wedge \omega + f \, d\omega.$$

By Lemma 3.3 then $d\omega = 0$. Now note that $df = \sum_\nu \frac{\partial f}{\partial z_\nu} dz_\nu$. Since ω has all its holomorphic differentials then we cannot add any more. It follows that $df \wedge \omega = 0$. Thus $d(f\omega) = 0$, and so $f\omega$ is a closed $(n, n-1)$ form. □

Now by Stokes' theorem applied to $D \setminus \overline{B(r)}$ we have that

$$\int_{bD} f\omega - \int_{S_r} f\omega = \int_{D \setminus \overline{B(r)}} f\omega = \int_{D \setminus \overline{B(r)}} d(f\omega) = 0$$

and so

$$\int_{bD} f\omega = \int_{S_r} f\omega = f(0) \frac{(2\pi i)^n}{(n-1)!} + \alpha(r)$$

where $\alpha(r) \to 0$ as $r \to 0$. This last equality is true as we can write

$$\begin{aligned}\int_{S_r} f\omega &= \int_{S_r} [f(0) + (f - f(0))] \, \omega \\ &= f(0) \int_{S_r} \omega + \int_{S_r} [f - f(0)] \, \omega\end{aligned}$$

where the first term is 0 by Lemma 3.3 and the second term can be written as $\alpha(r)$ where $\alpha(r) \to 0$. However $\int_{bD} f\omega$ does not depend on r, thus $\alpha(r) \equiv 0$. Thus

$$f(0) = \frac{(n-1)!}{(2\pi i)^n} \int_{\partial D} f\omega.$$

Replacing 0 with any $z \in D$, ie $\omega(\zeta) \to \omega(z-\zeta)$, gives us the so-called **Bochner-Martinelli ernel**

$$\omega_{BM}(\zeta - z) := \frac{(n-1)!}{(2\pi i)^n} \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}(\overline{\zeta}_\nu - \overline{z}_\nu)}{|\zeta - z|^{2n}} \, d\overline{\zeta}[\nu] \wedge d\zeta.$$

Theorem 3.12. *$\forall D \subset \mathbb{C}^n$ with bD piecewise smooth, $\forall f \in \mathcal{O}(D) \cap C(\overline{D})$, $\forall z \in D$, then*

$$f(z) = \int_{bD} f(\zeta)\omega_{BM}(\zeta - z). \tag{3.1}$$

Proof. This is a result of the previous lemmas. □

Remark.

- For $n = 1$ then

$$\omega_{BM}(\zeta - z) - \frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^2} \, d\zeta = \frac{d\zeta}{\zeta - z}$$

  which is the Cauchy kernel. Thus the Bochner-Martinelli integral formula is a generalization of the one-dimensional Cauchy integral formula. In some sense BM is better than the CIF as it uses the full boundary of D.

- If $z \in \mathbb{C}^n \setminus \overline{D}$ then since $f(\zeta)\omega_{BM}(\zeta - z)$ is closed in D we have that $\int_{bD} f(\zeta)\omega_{BM}(\zeta - z) = 0$.

- We have that

$$\int_{bD} \omega_{BM}(\zeta - z) = \begin{cases} 1 & z \in D, \\ 0 & z \in \mathbb{C}^n \setminus \overline{D}. \end{cases}$$

  Note that this is not defined for $z \in bD$.

- If f is only continuous on ∂D then (3.1) gives a harmonic function in D. This is like a higher dimensional Poisson kernel.

- BM has very weak requirements on D.

- In some sense BM is worse than CIF, as $\frac{1}{|\zeta - z|^{2n}}$ depends on $\overline{z}$, and so depends anti-holomorphically on z. For quite a long time people looked for alternatives that both produce homolorphic extensions and has no anti-holomorphic dependence.

We now want to use BM with stronger conditions on the behaviour on ∂D to get holomorphic functions, not just harmonic functions.

3.12 Analytic Continuation and Tangential CR Equations

If $\phi : \mathbb{C}^n \to \mathbb{R}$ is smooth with $\phi(0) = 0$ and $\nabla\phi(0) \neq 0$, then by the implicit function theorem then $\{\, \phi = 0 \,\}$ is a real manifold in $\mathbb{C}^n$ of real dimension $2n - 1$. This is a real hypersurface. The primary example is the boundary of a domain – such a ϕ is called a defining function of the hypersurface.

IMAGE

The normal direction at a point is $\mathit{span}\,\{\, \nabla\phi \,\} \cong \mathbb{R}$. This result can also be seen by looking at $\mathrm{d}\phi$.

Now let $\Phi : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function such that $\phi(0) = 0$ and $\mathrm{d}\phi(0) \neq 0$. Then $\{\, \phi(z) = 0 \,\}$ is a complex hypersurface. Then

$$\dim_{\mathbb{C}} \phi^{-1}(0) = n - 1,$$
$$\dim_{\mathbb{R}} \phi^{-1}(0) = 2n - 2.$$

It follows that complex hypersurfaces do not divide the ambient space.

Now let $M = \{\, \phi = 0 \,\}$ such that $\mathrm{d}\phi_M \neq 0$ be a real hypersurface. Let $p \in M$, and let T_pM be the real tangent space to M at p. Since the real dimension of the tangent space is $2n - 1$ there is a complex linear subspace $H_pM \subset T_pM$ such that $\dim_{\mathbb{C}} H_pM = n - 1$. In fact it is true that

$$H_pM = T_pM \cap iT_pM.$$

Let $f \in C^1(U)$, $S \subset U$ being a real hyperspace. We can write

$$\bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\, \mathrm{d}\bar{z}_j.$$

At a point $\zeta \in S$ we can represent $\bar{\partial}$ as a sum.

IMAGE

We can split $\bar{\partial}$ into $\bar{\partial}_N$ along the complex normal and $\bar{\partial}_T$ along the complex tangent and so write

$$\bar{\partial} f = \bar{\partial}_N f + \bar{\partial}_T f.$$

If ϕ is a defining function of S then the complex span of $\bar{\partial}\phi$ is the complex normal direction to $H_\zeta S$. Thus we have

$$\bar{\partial}_N f = \lambda \cdot \bar{\partial}\phi$$

for $\lambda \in \mathbb{C}$. It follows that

$$\bar{\partial}_T f = \bar{\partial} f - \bar{\partial}_N f.$$

Definition. $\bar{\partial}_T$ is called the **tangential CR operator on** S.

This is the $\bar{\partial}$ operator but "constrained to S".

Example 3.15. Let $S = \{x_n = 0\}$. $\phi = x_n = \frac{1}{2}(z_n + \bar{z}_n)$ is the defining function. Then

$$\bar{\partial}\phi = \frac{1}{2}\, \mathrm{d}\bar{z}_n.$$

The real normal is the $\mathbb{R}$-span of $(0, \cdots, 0, 1) \cong x_n$-axis. The complex normal is the $\mathbb{C}$-span of $(0, \cdots, 0, 1) = z_n$-axis.

ADD PIC ◁

This leads to the so-called **tangential CR equations**

$$\frac{\partial}{\partial \bar{z}_1}, \cdots, \frac{\partial}{\partial \bar{z}_{n-1}}.$$

Example 3.16. Let

$$S = \{2x_n + |z'|^2 = 0\}.$$

Note that S is biholomorphically equivalent to $b\mathbb{B}(0,1)$. Then

$$\bar{\partial}\phi = d\bar{z}_n + \sum_{j=1}^{n-1} z_j \, d\bar{z}_j.$$

Thus we have the complex normal direction

$$\bar{\partial}_N = \frac{\partial}{\partial \bar{z}_n} + \sum_{j=1}^{n-1} z_j \frac{\partial}{\partial \bar{z}_j}.$$

Then the complex tangent direction is

$$\bar{\partial}_T = \frac{\partial}{\partial \bar{z}_j} - \bar{z}_j \frac{\partial}{\partial z_n} \quad j = 1, \cdots, n-1.$$

◁

Exercise. Show $\bar{\partial}_T$ is tangent to $\bar{\partial}_N$.

Definition. Let $S \subset \mathbb{C}^n$ be a real hypersurface, and let $f \in C^1(S)$. f is called a **Cauchy-Riemann function** (ie a CR function) if f satisfies $\bar{\partial}_T f(\zeta) = 0$ for all $\zeta \in S$.

This definiteion makes sense. Indeed we have the following lemma.

Lemma 3.5. *If $U \subset \mathbb{R}^n$, $\phi \in \mathbb{C}^k(U)$, for some $k \geq 1$, with $\nabla\phi \neq 0$ and $\psi \in C^k(U)$ vanishes everywhere where ϕ is 0. Then there exists $h \in C^{k-1}(U)$ such that*

$$\psi(x) = h(x)\phi(x).$$

Proof. Showing such an h exists is easy. Showing $h \in C^{k-1}(U)$ is harder. □

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Suppose now that F, G are 2 different extensions of f from S to a neighbourhood $U \supset S$. Since $F - G = 0$ on S then by the lemma $F - G = h\phi$ where ϕ is the defining function of S and $h \in C^0(U)$. Then on S we have

$$\bar{\partial}F - \bar{\partial}G = h\bar{\partial}\phi.$$

This is since $\phi = 0$ on S. But then $\bar{\partial}\phi$ is a complex normal direction. This implies that

$$\bar{\partial}_T F - \bar{\partial}_T G = 0$$

as this "only sees" the normal direction. Thus $\bar{\partial}_T F = 0$ is independent of the existence of F.

Theorem 3.13. *The tangential CR equations*

$$\bar{\partial}_T f = 0 \tag{3.2}$$

at $\zeta \in S$ are equivalent to

- $\bar{\partial} f(\zeta)$