A Universality Theorem

Not at all Combinatorics or Optimization

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Weierstrass

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Theorem (Weierstrass (1885))

Fix an interval $K = [a, b] \subset \mathbb{R}$*, and an* $\epsilon > 0$*. Then for all* $f : \mathbb{R} \to \mathbb{R}$ *that are continuous on* K *there exists a polynomial* $g(x) = g_{K,\epsilon,f}(x)$ *such that*

$$\left| f(x) - g(x) \right| < \epsilon \qquad \forall x \in K.$$

Moreover this theorem has a constructive proof.

Weierstrass Example

Example

Define

$$K = [1, 2]$$

$$\epsilon = 0.5$$

$$f = e^{-x^2}$$

Then the polynomial $g(x) = 0.7 + x - 3.25x^2 + 2.5x^3 - 0.8x^4 + 0.1x^5$ satisfies $|f(x) - g(x)| < \epsilon$ for $x \in K = [1, 2]$.

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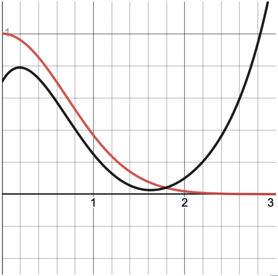

Weierstrass

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Theorem (Stone-Weierstrass (1937))

Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X, \mathbb{R})$ which contains a non-zero constant function. Then A is dense in $C(X, \mathbb{R})$ if and only if it separates points.

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Funny-looking, but implies Weierstrass.

Weierstrass Generalizations

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Uniform approximation $\longrightarrow$ stronger approximation?

For us:

$$f : \mathbb{R} \to \mathbb{R} \longrightarrow f : \mathbb{C} \to \mathbb{C}.$$

What is complex analysis?

How does complex analysis differ from real analysis?

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Real analysis

$$f : \mathbb{R}^k \to \mathbb{R}^k$$

– Simple results, “easy” to be differentiable

Complex analysis

$$f : \mathbb{R}^{2k} \to \mathbb{R}^{2k} \text{ with extra structure}$$

– Powerful results, “hard” to be differentiable

Poles

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Example

$f(z) = \frac{1}{(z+2)(z-1)^2}$ has poles at $z = -2$ and $z = 1$.

Analyticity

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We say a function $f : \Omega \to \mathbb{C}$ is **analytic at** $z_0 \in \Omega$ if we can write

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e^z, $sin(z)$, $cos(z)$ – analytic everywhere in $\mathbb{C}$, called **entire**.

Example (Not Entire)

$\Gamma(z)$ – analytic at most places in $\mathbb{C}$, has poles at $\{0, -1, -2, \cdots\}$.

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Fix a compact set $K \subset \mathbb{C}$, an $\epsilon > 0$, and let $A \subset \mathbb{C}$ be a finite set with at least one point in each component of K^c. Then for all f analytic on K there exists a rational function $R(z) = R_{K,\epsilon,A,f}(z)$ such that

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Non-constructive proof.

Mergelyan's Theorem

Corollary

Fix a compact set $K \subset \mathbb{C}$ with K^c connected, and an $\epsilon > 0$. Then for all f analytic on K there exists a polynomial $P(z) = P_{K,\epsilon,f}(z)$ such that

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Theorem (Mergelyan (1951))

Fix a compact set $K \subset \mathbb{C}$ with K^c connected, and an $\epsilon > 0$. Then for all f analytic on K° and continuous on K there exists a polynomial $P(z) = P_{K,\epsilon,f}(z)$ such that

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Constructive proof.

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Mergelyan (with a little extra work) states that

Theorem (Mergelyan v2)

Let $K \subset \mathbb{C}$ be compact. Then

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What about the other containment signs?

AC Capacity

Definition (Continuous Analytic Capactity (not really))

Let $M \subset \mathbb{C}$ be compact. Then there is a property of M called the **continuous analytic capacity of** M. This is written $\alpha(M) \in [0, \infty)$.

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The actual definition is not very enlightening, but it is a computable/estimable positive real number associated with the set. Broadly the "complexity" of a set.

Vitushkin's Theorem

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Theorem (Vitushkin (1966))

Let $K \subset \mathbb{C}$ be compact. Then

$$\alpha(D \setminus K) = \alpha(D \setminus K^\circ) \ \forall D \text{ open disk}$$
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Beautifully characterizes when $R(K) = A(K)$!

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3. My favourite theorem

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Fact (almost)

The Riemann Zeta function $\zeta(z)$ is a universal function!

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Takeaways in descending importance to your life:

1. Zeta function universality $=$ power/beauty/complexity of ζ.
2. Existence of universal function $=$ beauty of complex analysis.
3. Approximation is a tool that exists.

Continuous Analytic Capacity

Definition (Continuous Analytic Capacity)

Let $M \subset \mathbb{C}$. Define $\mathscr{C}(M)$ to be the set of all functions $f : \mathbb{C} \to \mathbb{C}$ st

1. $f(z) \to 0$ as $z \to \infty$,
2. $|f(z)| \leq 1$ for all $z \in \mathbb{C}$,
3. f is analytic on K^c for some compact $K \subset M$,
4. f is continuous on $\mathbb{C}$.

By these conditions we can expand f at "∞" as

$$f(z) = \frac{c_1^f}{z} + \frac{c_2^f}{z^2} + \frac{c_3^f}{z^3} + \cdots .$$

Then we define the **continuous analytic capacity** of M as

$$\alpha(M) := \sup_{f \in \mathscr{C}(M)} |c_1^f| = \sup_{f \in \mathscr{C}(M)} |\lim_{z \to \infty} z f(z)| = \sup_{f \in \mathscr{C}(M)} |f'(\infty)|.$$

Riemann Zeta Universality

The actual result is as follows.

Theorem (Voronin's Universality Theorem (1975))

Let $0 < r < \frac{1}{4}$*. Let* $f(z)$ *be a function that is analytic and* **non-zero** *in the disk* $D_r(0)$ *and continuous on* $\overline{D_r(0)}$*. Then there is some* $\tau \in \mathbb{R}$ *such that*

$$|\zeta(z + \tfrac{3}{4} + i\tau) - f(z)| \quad \forall z \in D_r(0).$$

Corollary

Same result, but with any arbitrary open disk $D_r(\omega)$ *with* $r > 0$ *instead of small ones centered at the origin, permitting a single affine shift* $z \mapsto \frac{z-\omega}{8r}$ *to get our disk back to the origin.*