Asymptotics of Lattice Walks

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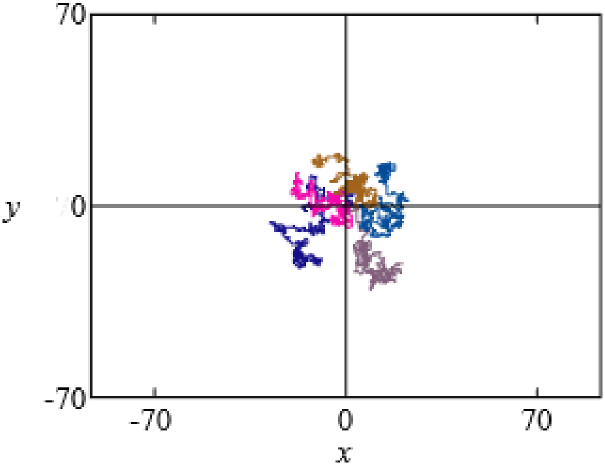
Dec 2023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Generating Functions

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We can use this framework to count lattice walks!

Lattice Walks

Definition (Lattice Walk Model)

A d-dimensional **lattice walk model** consists of:

- a **step set** $S \subseteq \mathbb{Z}^d$,
- a **restricting region** $R \subseteq \mathbb{Z}^d$,
- a **starting point** $p \in R$,
- a **terminal set** $T \subseteq R$.

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We always let $p = 0$ and $T = R$.

Lattice Walks

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Reformulated in generating function language:

Main Question

Given a step set S and a restricting region R, let s_n be the number of lattice walks of length n using S and staying in R.

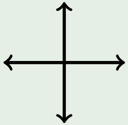
What is the coefficient of z^n in $S(z) = \sum_{n \geq 0} s_n z^n$?

Unrestricted Walks

If $R = \mathbb{Z}^d$ this is easy.

Unrestricted NSEW steps

Let $S = \{(\pm 1, 0), (0, \pm 1)\}$ and $R = \mathbb{Z}^2$.



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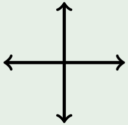

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In general:

Unrestricted Walks

There are $|S|^n$ unrestricted lattice walks of length n using S.

Halfspace Walks

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This means finding asymptotics for s_n is automatic.

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This is too hard in general. How do we fix this? Restricting S.

Highly Symmetric Step Sets

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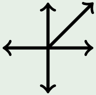
Example

If $S = \{(\pm 1, 0), (0, \pm 1), (1,1)\}$



$$S(z_1, z_2) = z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2$$

Highly Symmetric Step Sets

We also enforce some amount of *symmetry*.

Definition (Highly Symmetric Step Sets)

A step set S is **highly symmetric** if

$$S(z_1, \cdots, z_j, \cdots, z_d) = S\left(z_1, \cdots, \frac{1}{z_j}, \cdots, z_d\right)$$

for all $1 \leq j \leq d$.

Highly Symmetric Step Sets

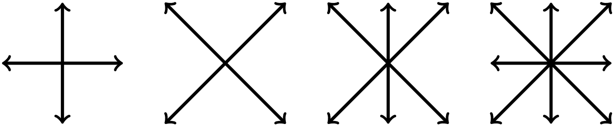
Asymptotics for highly symmetric step sets is known.

Theorem (Melczer Mishna 2016 [MM16])

Suppose S is a highly symmetric step set. If s_n is the number of walks of length n using S that remains in $\mathbb{N}^d$ then

$$s_n \sim \left[\frac{|S|^{d/2}}{\pi^{d/2}(a_1 \cdots a_d)^{1/2}}\right] \cdot \frac{|S|^n}{n^{d/2}},$$

where a_j is the number of steps in S that have j-th coordinate 1.



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for all $1 \leq j < d$.

This means we can write

$$S(\mathbf{z}) = \frac{1}{z_d} A(z_1, \cdots, z_{d-1}) + Q(z_1, \cdots, z_{d-1}) + z_d B(z_1, \cdots, z_{d-1})$$

where A, Q, B are all highly symmetric.

Mostly Symmetric Step Sets

Asymptotics depend on what the average direction of a step is.

Definition (Drift)

We say that

- S has **positive drift** if $B(\mathbf{1}) - A(\mathbf{1}) > 0$,
- S has **negative drift** if $B(\mathbf{1}) - A(\mathbf{1}) < 0$,
- S has **zero drift** if $B(\mathbf{1}) - A(\mathbf{1}) = 0$.

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Asymptotics for the positive and negative case are known.
Asymptotics for the zero drift case is *open*.

Positive Drift Asymptotics

Theorem (Melczer Wilson [MW19])

If S is mostly symmetric with positive drift then

$$s_n \sim \left[\left(1 - \frac{A(1)}{B(1)}\right) \frac{S(1)^{d/2}}{(2\pi)^{d/2}} \cdot \frac{1}{(a_1 \cdots a_d)^{1/2}} \right] \cdot \frac{S(1)^n}{n^{d/2-1/2}}$$

Negative Drift Asymptotics

Theorem (Melczer Wilson [MW19])

If S is mostly symmetric with negative drift and $Q(z) \neq 0$ then

$$s_n \sim C_\rho \cdot \frac{S(1,\rho)^n}{n^{d/2+1}}.$$

If S is mostly symmetric with negative drift and $Q(z) = 0$ then

$$s_n \sim C_\rho \cdot \frac{S(1,\rho)^n}{n^{d/2+1}} + C_{-\rho} \cdot \frac{S(1,-\rho)^n}{n^{d/2+1}}.$$

Where ρ, C_ρ, and $C_{-\rho}$ are explicit constants.

Proof Overview

How is this done?
Through **analytic combinatorics in several variables** (ACSV).

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- Approximate integral using saddle-point method.

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At the singularity determining asymptotics both the denominator and numerator vanish. Other pathologies give an integral of the form

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Now $\sim$ 4 years later we have a better idea of how to approximate.

Zero Drift Asymptotics

Theorem (K. Melczer [KMon])

Suppose S is a mostly symmetric step set with zero drift. Let s_n be the number of walks of length n using S that remain in $\mathbb{N}^d$. Then

$$s_n \sim \left[\frac{|S|^{d/2}}{\pi^{d/2}(a_1 \cdots a_d)^{1/2}} \right] \cdot \frac{|S|^n}{n^{d/2}}$$

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This (sort of) finishes off all walks that can be done with ACSV.

Conclusion

Thank you for listening.

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